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Non-critical supergravity ($d > 1$) and holography

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Abstract: In this paper we investigate the supergravity equations of motion associated with non-critical ($d > 1$) type-II string theories that incorporate RR forms. Using a superpotential formalism we determine several classes of solutions. In particular we find analytic backgrounds with a structure of $AdS_{p+2} \times S^{d-p-2}$ and numerical solutions that asymptote a linear dilaton with a topology of $R^{1,d-3} \times R \times S^1$. The SUGRA solutions we have found can serve as anti holographic descriptions of gauge theories in a large-N limit which is different than the one of the critical gauge/gravity duality. It is characterized by $N \to \infty$ and $g^2_{\rm YM} N \sim 1$. We have made the first steps in analyzing the corresponding gauge theory properties like Wilson loops and the glue-ball spectra.

Keywords: AdS-CFT and dS-CFT Correspondence, Supergravity Models.

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Contents

1. Introduction

Gauge/gravity holographic duality has been proven to be a powerful tool of studying strongly coupled gauge dynamics. However, the anti holographic picture of hadronic physics suffers from several limitations.

One such limitation, which is shared by all confining gauge theories duals of critical supergravity (SUGRA) backgrounds, is the fact that generically their spectrum includes on top of the ordinary hadronic states, Kaluza Klein states of the same scale of mass as that of the hadrons. So far we do not know of a mechanism to disentangle the desired hadronic particles from the KK contamination.

One obvious way to overcome this problem is to abandon string theories in critical dimensions and their corresponding SUGRA backgrounds, and instead study the holographic duality with non critical string theories and SUGRA models. Needless to say that superstring theories of dimensions close to four are desired not only as candidates of the theory of gauge dynamics but also of the theory of quantum gravity and grand unification.

Non-critical string theories in d dimensional space-time are characterized by having a Liuville mode on their world-sheet which combines together with additional $d-1$ coordinates. For $d \leq 2$ these theories have been shown to be consistent and it was believed that one cannot pass the so called "$c = 1$ barrier" (for bosonic strings). However, in [1, 2] it was shown that non-critical superstring theories for $d > 2$ that admit supersymmetry in space time are consistent. An explicit construction of world sheet theory with $\mathcal{N} = (2, 2)$ supersymmetry which includes an $\mathcal{N} = 2$ super Liuville was proven to have in even dimensions, space time supersymmetry. The corresponding GSO projection eliminates the tachyons and by that guarantees that the non-critical superstring theories are consistent. It was then conjectured that the super Liuville theory is equivalent to the CFT describing the euclidean black hole which is nothing but the CFT on a cigar. The corresponding $\mathcal{N} = 2$ case was discussed in [3, 4, 5] and in [6]. In [3, 4, 5] it was argued that certain non-critical superstring theories follow a double scaling limit of critical ten dimensional superstring theories on CY manifold with an isolated singularity.

Non-critical type-II SUGRA backgrounds are believed to be solutions of the equations of motion associated with the low energy effective action of the type-II non-critical superstring theory. Recall that the equations of motion are in fact the β function equations associated with the world sheet scale invariance so that every solution of them is guaranteed to relate to scale invariant vacuum of the corresponding string theory.

The idea to extend the anti-holographic description of gauge theories to non-critical SUGRA backgrounds was introduced first in [7] where a proposal of a dual of pure YM in terms of a 5d non-critical gravity background was proposed. Following this idea there were several attempts to find solutions of the non-critical effective action that are adequate as duals of gauge theories [8]–[18]. For instance in [16] certain five dimensional non-critical type 0 backgrounds were written down and where shown to admit certain properties of non-supersymmetric Yang Mills theory in four dimensions. In contrast to the $d < 10$ theories, the non-critical supergravity with $d \gg 10$ exhibits new features [19] like dilaton potentials with non-trivial minima at arbitrarily small cosmological constant and d-dimensional string coupling.

From the study of the gravity duals of non-trivial gauge theories, mainly the critical ones but also the non-critical ones, it has become clear that a radial direction of the bulk theory plays the role of the re-normalization scale of the related gauge theory. That was the rational in [7] to look for a five dimensional setup dual to the pure YM theory. For theories with supersymmetries one has to incorporate also isometries of the background that will correspond to R symmetries and global symmetries of the dual gauge theories. For example the classical $\mathrm{U}(1)_R$ symmetry of the $\mathcal{N} = 1$ SYM may follow from an isometry of an additional S^1 part of the d dimensional space time hence one would anticipate based on this argument, that a six dimensional space time may be the "minimal" dual of $\mathcal{N} = 1$ SYM.

The aim of this paper is to search for solutions of the equations of motion of the low energy effective action associated with non critical type-II string theories. In particular our goal is to look for solutions that may serve as useful anti-holographic descriptions of gauge theories. Rather than solving the second order equations of motion we transform the problem to that of a set of BPS equations derived from a superpotential that relates to the potential of the system.[1] The non-criticality nature is manifested by a term in the potential that is proportional to the non-critical factor $c = 10 - d$. In the formulation that we are using this contribution to the potential takes the form

$$V_{\text{non-critical}} = \frac{c}{2} e^{2(n\lambda + k\nu - 2\phi)} \tag{1.1}$$

where ϕ is the dilaton, and the fields $e^{2\lambda}$ and $e^{2\nu}$ are the wrap factors of the world volume and compact transverse part of the metric respectively (see (2.1)).

We found several families of solutions. With no RR fields we find the solutions of the cylinder with linear dilaton, the cigar its T-dual the trumpet solution and certain generalizations of them.[2] One class of solutions with RR fields are of the form $AdS_{n+1} \times S^k$ with $k \neq 1$, $n+1 \neq k$ and where the total dimension is $d = n+k+1$. Close relatives of these solutions are near extremal solutions based on these $AdS_{n+1} \times S^k$ solution. We re-derive using the superpotential approach the charged black hole solution of [24] including its near horizon AdS_2 solution as well as their T-dual solution. As mentioned above backgrounds that incorporate a cylindrical geometry and linear dilaton [25] has special interest both from the point of view of the world sheet description of non-critical superstring theory as well as from the point of view of backgrounds that correspond to brane configurations associated with interesting gauge theories. We therefore put a special emphasis in looking for solutions that admit this geometry. Indeed we found solutions that asymptote $R^{1,d-3} \times R \times S^1$ and

[1] See i.g. [20, 21, 22], where the superpotential approach has been extensively used.

[2] See [23] for a review of T-duality.

incorporate RR fields. We present the corresponding numerical solutions in the full range of the radial direction and the approximated analytic behavior both asymptotically as well as close to the origin of the radial direction. In this paper we will not discuss the exact superstring solutions in non-trivial non-critical space-times with NS-NS and no RR fields based on superconformal world-sheet supersymmetry (see [26] for a review).

A generic feature of the solution is that due to the non-criticality factor the scalar curvature of various background is not small but rather is proportional to $c = 10 - d$. Therefore neglecting higher curvature terms is not really justified. The conjecture is that for special solutions like the $AdS_{n+1} \times S^k$, the structure of the background would not be modified by the higher order curvature contributions but rather only the corresponding radii will be corrected.

For the classes of solutions we have discovered, we analyze the holographic boundary field theories. We substantiate the idea that the boundary field theory is a gauge theory by proving that the Bekenstein Hawking entropy of the SUGRA background scales like N^2, where N is the flux of the RR form. We show that the anti-holographic SUGRA picture of the gauge field corresponds to a large-N limit where

$$N \to \infty \qquad g_{\rm YM}^2 N \sim 1\,, \tag{1.2}$$

which is between perturbative large-N limit and the one used in the gauge/gravity duality. We briefly discuss the Wilson loop and the glue-ball spectrum for duals of pure YM in 3 and 4 dimensions that follow from the large temperature limit of the near extremal $AdS_{n+1} \times S^k$ solutions. We also address the these properties for the theories, which may be viewed as the RR deformation of the cigar solution.

The paper is organized as follows: in the next section we present the general setting, namely the non-critical low energy effective action, the corresponding equations of motion and the superpotential approach that leads to BPS equations compatible with the equations of motion. Section 3 is a warmup exercise where we find the D_p solutions by switching off the non-criticality term. Section 4 is devoted to non-critical backgrounds that do not include NS or RR forms. In subsection 4.1 we present solutions of backgrounds with no transverse directions, In subsection 4.2 we derive the cylinder solution, the cigar solution and certain generalization of it. The next subsection describes the cigar and trumpet solutions as T-duals and in the last subsection we find a special solution in four dimensional compact transverse space. Section 5 is devoted to the conformal $AdS_{n+1} \times S^k$ backgrounds both as solutions of the equations of motion as well as solutions of the BPS equations. In subsection 5.2 we describe the AdS black hole associated with the $AdS_{n+1} \times S^k$ backgrounds. In section 6 we describe two dimensional solutions that incorporate RR fields. In particular we derive the charged black hole solution of [24], take the near horizon limit that leads to the AdS_2 solution, and construct the T-dual of the charge black hole solution. Section 7 is devoted to solutions that include RR forms in the cylindrical geometry or differently stated RR deformations of the cigar/trumpet solutions. We describe numerically three families of solutions. We derive approximate expressions for these backgrounds both asymptotically as well as close to the origin. A brief discussion of backgrounds that include NS three form and no RR forms is presented in section 8. In section 9 we explore the properties of the

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holographic dual gauge theories associated with the background solutions we had found. We first show that the entropy of the non-critical SUGRA backgrounds admit entropy that scales like N^2. We then show in 9.2 that the non-critical duality in fact holds for a novel large-N limit where $g_{\rm YM}^2 N \sim 1$. We then describe in 9.3 the gauge theories which are the cousins of the $AdS_{n+1} \times S^k$ backgrounds. In 9.4 we analyze the dual gauge theory of the AdS black hole solutions in particular the Wilson loops and the glue-ball spectra are addressed. The duals of the RR deformed cylinder SUGRA backgrounds are described in subsection 9.5. To the benefit of the reader we include several appendices with certain explicit computations of the quantum mechanical effective action, The BPS equations both in the string an Einstein frames, certain potential — superpotential relations, the derivation of the cigar solution in the Einstein frame and the solution of [24] from the equations of motion.

2. General setting

The metric in the *string* frame is taken to depend only on the radial coordinate τ. It takes the form

$$l_s^{-2} ds^2 = d\tau^2 + e^{2\lambda(\tau)} dx_{||}^2 + e^{2\nu(\tau)} d\Omega_k^2 \tag{2.1}$$

where $dx_{||}^2$ is n dimensional flat metric, and $d\Omega_k^2$ is a k dimensional sphere.

The bosonic part of the non-critical SUGRA action in d dimensions takes the form

$$\begin{aligned} S = \int d^{n+k+1}x \sqrt{G}\, e^{-2\phi} \left(R + 4(\partial\phi)^2 + \frac{c}{\alpha'}\right) - \\ - \frac{e^{-2\phi}}{2} \int H_{(3)} \wedge \star H_{(3)} - \sum_p \frac{1}{2} \int F_{(p+2)} \wedge \star F_{(p+2)}\,, \end{aligned} \tag{2.2}$$

where

$$\frac{c}{\alpha'} = \frac{10-d}{\alpha'} \tag{2.3}$$

is the non-criticality central charge term.[3] The latter term corresponds to the deviation of the dilaton beta function at non-critical space-time dimension from the one at critical dimension. F_{p+2} is a RR form that corresponds to a D_p brane with $n = p+1$ dimensional world volume, and $H_{(3)}$ is the NS three form. Throughout this paper we will investigate solutions with only one non trivial RR form.

Upon substituting the metric (2.1) into the action and performing the integration one finds [8] (see appendix A for the derivation):

$$S = l_s^{-2} \int d\rho \left(\left[-n(\lambda')^2 - k(\nu')^2 + (\varphi')^2 + ce^{-2\varphi} + (k-1)ke^{-2\nu-2\varphi} \right] \right) + S_{\rm RR} + S_{\rm NS}\,, \tag{2.4}$$

where $d\tau = -e^{-\varphi} d\rho$, $(A)' = \partial_\rho A$ and

$$\varphi = 2\phi - n\lambda - k\nu \tag{2.5}$$

is the "shifted" dilaton.

[3] Throughout this paper c is a dimensionless parameter ($c = 10 - d$). Also all the coordinates on the r.h.s. of (2.1) are dimensionless in accordance with the l_s^{-2} factor on the l.h.s.

Assuming that the RR form also depends only on the radial direction, namely, $F = \partial_\tau A dx^0 \wedge \cdots \wedge dx^p \wedge d\tau$, the RR part of the action reads

$$S_{\rm RR} = -\int d\rho \left(\frac{1}{4} e^{-n\lambda + k\nu + \varphi}(A')^2\right) = -Q^2 \int d\rho e^{n\lambda - k\nu - \varphi}\,, \tag{2.6}$$

where we made the substitution $A' = 2Qe^{n\lambda - k\nu - \varphi}$ which is the solution of the equation of motion of $A'' - A'(n\lambda' - k\nu' - \varphi') = 0$. In fact the assumption was that the Hodge dual of the RR form is proportional to the volume form ω_k of the transversal S^k, namely $\star F_{(p+2)} = N\omega_k$, where $p = n-1$ and N is the number of the Dp-branes. Q is equal N up to some numerical factor coming, in particular, from the integration over the S^k part of the metric. Alternatively we can assume that $F_{(p+2)} = N\omega_k$. This way we will end up with a stack of Dp, where this time $p = k-2$. For instance, for $k = 1$ and $n = 4$ the corresponding form is $F_{(5)}$ so we take its Hodge dual $F_{(1)} \sim d\theta$ (the 0-form potential $A_{(0)}$ depends linearly on θ) where θ is the coordinate on the S^1. Alternatively, this can be viewed as $F_{(5)} = \star_6 F_{(1)} \sim \star d\theta$ ($A_{(4)}$ is a function of τ). In the former case this is interpreted as a stack of $D3$-branes and in the later case we have "magnetic" duals, namely $D(-1)$-branes. In both cases the form contribution (2.6) to the action is the same.

Similarly, for $k = 3$ the NS 3-form is given by $H_{(3)} = N\omega_3$ with N denoting the number of the NS5 branes. Under these assumptions upon substituting the solution of the equation of motion for the NS forms, the NS action reads

$$S_{\rm NS} = -Q^2 \int d\rho e^{-2k\nu - 2\varphi}\,. \tag{2.7}$$

The second order equations of motion are:

$$\begin{aligned} \partial_\rho^2 \lambda - \frac{1}{2} Q_{\rm RR}^2 e^{n\lambda - k\nu - \varphi} &= 0\,, \\ \partial_\rho^2 \nu - (k-1)e^{-2\nu - 2\varphi} + \frac{1}{2} Q_{\rm RR}^2 e^{n\lambda - k\nu - \varphi} + Q_{\rm NS}^2 e^{-2k\nu - 2\varphi} &= 0\,, \\ \partial_\rho^2 \varphi + (c + (k-1)ke^{-2\nu})e^{-2\varphi} - \frac{1}{2} Q_{\rm RR}^2 e^{n\lambda - k\nu - \varphi} - Q_{\rm NS}^2 e^{-2k\nu - 2\varphi} &= 0\,. \end{aligned} \tag{2.8}$$

In terms of λ,ν and the dilaton ϕ it takes the form

$$\begin{aligned} \partial_\rho^2 \lambda - \frac{1}{2} Q_{\rm RR}^2 e^{2(n\lambda - \phi)} &= 0\,, \\ \partial_\rho^2 \nu - (k-1)e^{2(n\lambda + (k-1)\nu - 2\phi)} + \frac{1}{2} Q_{\rm RR}^2 e^{2(n\lambda - \phi)} + Q_{\rm NS}^2 e^{2(n\lambda - 2\phi)} &= 0\,, \\ \partial_\rho^2 \phi + \frac{c}{2} e^{2(n\lambda + k\nu - 2\phi)} - \frac{(n+1-k)}{4} Q_{\rm RR}^2 e^{2(n\lambda - \phi)} + \frac{(k-1)}{2} Q_{\rm NS}^2 e^{2(n\lambda - 2\phi)} &= 0\,. \end{aligned} \tag{2.9}$$

Again the NS term is relevant only for $k = 3$. Solving these equations one also has to imply the zero-energy constraint, in other words the hamiltonian derived from the 1d action (2.4) must be set to zero. This constraint which is in fact the equation of motion associated with $g_{\tau\tau}$ takes the following form

$$n(\partial_\tau \lambda)^2 + k(\partial_\tau \nu)^2 - (\partial_\tau \varphi)^2 + c + (k-1)ke^{-2\nu} - Q_{\rm RR}^2 e^{n\lambda - k\nu + \varphi} - Q_{\rm NS}^2 e^{-2k\nu} = 0\,. \tag{2.10}$$

2.1 The superpotential and BPS equations

It is well known that for supersymmetric backgrounds one can avoid the hurdle of solving second order differential equations and instead solve first order BPS equations. The solutions of the latter equations are also solutions of the equations of motion.

Consider the following general form of a background action

$$S = \int d\rho \left(-\frac{1}{2} G_{ab} f^{a\prime} f^{b\prime} - V(f) \right) . \tag{2.11}$$

The corresponding equations of motion are given by

$$\begin{aligned} 0 &= (G_{ab} f^{b\prime})' - \frac{1}{2}\partial_a G_{bc} f^{b\prime} f^{c\prime} - \partial_a V \\ &= G_{ab} f^{b\prime\prime} + (\partial_c G_{ab} - \frac{1}{2}\partial_a G_{bc}) f^{b\prime} f^{c\prime} - \partial_a V \\ &= G_{ab}(f^{b\prime\prime} + \Gamma^b_{cd} f^{c\prime} f^{d\prime} - \partial^b V) . \end{aligned} \tag{2.12}$$

and the zero-energy constraint reads $-\frac{1}{2} G_{ab} f^{a\prime} f^{b\prime} + V(f) = 0$.

If the potential is related in the following way to a superpotential W

$$V = \frac{1}{8} G^{ab} \partial_a W \partial_b W , \tag{2.13}$$

then the BPS equations are

$$f^{a\prime} = \frac{1}{2} G^{ab} \partial_b W. \tag{2.14}$$

It is straightforward to check that the BPS equations are compatible with the equations of motion (see appendix B) and with the zero energy condition. Applying this structure to the action of (2.4) we have that

$$G_{\lambda\lambda} = 2n \qquad G_{\nu\nu} = 2k \qquad G_{\varphi\varphi} = -2 \tag{2.15}$$

and

$$V = Q^2 e^{n\lambda - k\nu - \varphi} - (c + (k-1) k e^{-2\nu}) e^{-2\varphi} . \tag{2.16}$$

and therefore the relation between the potential and the superpotential reads

$$\frac{1}{n}(\partial_\lambda W)^2 + \frac{1}{k}(\partial_\nu W)^2 - (\partial_\varphi W)^2 = 16V \tag{2.17}$$

and the BPS equations are

$$\lambda' = \frac{1}{4n}\partial_\lambda W , \qquad \nu' = \frac{1}{4k}\partial_\nu W , \qquad \varphi' = -\frac{1}{4}\partial_\varphi W . \tag{2.18}$$

Another useful parameterization of the superpotential approach is the following. Consider the action that takes the following form

$$S = \int du\, e^{4A} \left(3(A')^2 - \frac{1}{2} G_{ab} f^{a\prime} f^{b\prime} - V(f) \right) , \tag{2.19}$$

where now $'$ denotes derivative with respect to u. In this formulation we look for a corresponding W which is related to the potential as follows

$$V = \frac{1}{8} G^{ab} \partial_a W \partial_b W - \frac{1}{3} W^2 \tag{2.20}$$

and the BPS equations are

$$f^{a\prime} = \frac{1}{2} G^{ab} \partial_b W \qquad A' = -\frac{1}{3} W(f)\,. \tag{2.21}$$

This formulation is adequate for the action in the Einstein frame. The reformulation of the whole background in the Einstein frame and the relation between the fields in the string frame to those in the Einstein frame are presented in appendix C.

3. Dp brane solutions in critical dimension ($c = 0$, $d = 10$)

Let us now as a warmup exercise apply the formalism of the superpotential and the BPS first order differential equations for the case of critical superstring, namely for ten dimensional space time. For the supergravity background with a RR form and no NS form, the superpotential equation has a simple solution:

$$W = 2\sqrt{2}\, Q e^{\frac{1}{2}(n\lambda - k\nu - \varphi)} - 4k e^{-\nu-\varphi}\,. \tag{3.1}$$

Let us write the BPS equations in terms of a radial coordinate r defined by

$$e^{-\varphi} d\rho = e^{\nu} \frac{dr}{r}\,. \tag{3.2}$$

We get:

$$\begin{aligned} r\frac{d\lambda}{dr} &= \frac{\sqrt{2}}{4} Q e^{\frac{1}{2}(n\lambda-(k-2)\nu+\varphi)} \\ r\frac{d\nu}{dr} &= -\frac{\sqrt{2}}{4} Q e^{\frac{1}{2}(n\lambda-(k-2)\nu+\varphi)} + 1 \\ r\frac{d\varphi}{dr} &= \frac{\sqrt{2}}{4} Q e^{\frac{1}{2}(n\lambda-(k-2)\nu+\varphi)} - k\,. \end{aligned} \tag{3.3}$$

Due to the fact that the exponential factor in all these equations is the same it is useful to define $\alpha = n\lambda - (k-2)\nu + \varphi$ which obeys the following equation

$$r\frac{d\alpha}{dr} = 2\sqrt{2}\, Q e^{\alpha/2} - 2(k-1)\,, \tag{3.4}$$

which is solved by:

$$e^{-\alpha/2} = g_s^{-1} r^{k-1} + \frac{\sqrt{2}\, Q}{k-1}\,. \tag{3.5}$$

Plugging this into the equations for λ and ν we find the expected result of the Dp-brane metric $ds^2 = e^{2\lambda} dx_\mu^2 + r^{-2} e^{2\nu}(dr^2 + r^2 d\Omega^2)$, where:

$$e^{2\lambda} = \left(1 + \frac{\frac{\sqrt{2}}{7-p} g_s Q}{r^{7-p}}\right)^{-1/2} \qquad \text{and} \qquad e^{2\nu} = r^2 \left(1 + \frac{\frac{\sqrt{2}}{7-p} g_s Q}{r^{7-p}}\right)^{1/2}, \tag{3.6}$$

where we have replaced $k-1=(10-n-1)-1=7-p$. The string coupling constant g_s is a free parameter of the solution and is set to be equal to an asymptotic value of the dilaton: $e^{\phi}|_{r\to\infty}=g_s$.

Using this result we can calibrate for the critical case the value of Q by comparing to the metric of the Dp brane. The result is

$$Q=\frac{7-p}{\sqrt{2}}2^{7-2p}\pi^{\frac{2-3p}{2}}\Gamma\left(\frac{7-p}{2}\right)N. \tag{3.7}$$

4. Solutions with zero RR and NS-NS charges ($Q=0$)

With no RR or NS-NS forms the potential include the non-critical term as well as a curvature term from the S_k part of the metric. There is no contribution form the curvature for $k=0$ and $k=1$ as can be seen from

$$V=-k(k-1)e^{-2\nu-2\varphi}-ce^{-2\varphi}\,. \tag{4.1}$$

The corresponding superpotential still has to solve (2.17) which can be simplified due to the $e^{-2\varphi}$ factor which is common to both terms of the potential. This obviously calls for an ansatz of the form

$$W(\lambda,\varphi,\nu)=4e^{-\varphi}w(\lambda,\nu) \tag{4.2}$$

for which (2.17) takes the form

$$\frac{1}{n}(\partial_\lambda w)^2+\frac{1}{k}(\partial_\nu w)^2-w^2=-k(k-1)e^{-2\nu}-c\,. \tag{4.3}$$

Due to the absence of the curvature part, the cases of no transverse S_k and of S_1 are special so we start with analyzing them first.

4.1 No transverse sphere ($k=0$)

In this case the metric is

$$l_s^{-2}ds^2=e^{2\lambda}dx_{\|}^2+d\tau^2\,. \tag{4.4}$$

The equation (4.3) for the superpotential is solved by

$$w(\lambda)=\pm\sqrt{c}\qquad\text{or}\qquad w(\lambda)=\pm\sqrt{c}\cosh(\sqrt{n}\,\lambda)\,. \tag{4.5}$$

For the first type of solution we end up with the following BPS equations

$$\lambda'=0\qquad\varphi'=\pm\sqrt{c}\,e^{-\varphi}\,. \tag{4.6}$$

With no loss of generality we can set $\lambda=0$ so the background includes a $(n+1)$ dimensional flat minkowskian metric

$$l_s^{-2}ds^2=-dt^2+\cdots+dx_{n-1}^2+d\tau^2 \tag{4.7}$$

and a linear dilaton

$$e^{\varphi}=\pm\sqrt{c}\,\rho\quad\to\quad\varphi=\pm\sqrt{c}\,\tau\quad\to\quad\phi=\pm\frac{\sqrt{c}}{2}\tau\,. \tag{4.8}$$

Note that in 10d the dilaton becomes constant and the ordinary ten dimensional flat Minkowski solution is retrieved. In fact the background with the linear dilaton corresponds to an exact 2d conformal theory on the world-sheet [27].

For the second type of superpotential, namely, $w(\lambda) = \pm\sqrt{c}\cosh(\sqrt{n}\,\lambda)$ the BPS equations take the following form

$$\lambda' = \sqrt{\frac{c}{n}}\,e^{-\varphi}\sinh(\sqrt{n}\,\lambda), \qquad \varphi' = \sqrt{c}\,e^{-\varphi}\cosh(\sqrt{n}\,\lambda)\,. \tag{4.9}$$

By transforming to derivatives with respect to τ these equations can be solved analytically. Instead we will now analyze the equations of motion and then come back to the solution of the BPS equations.

Indeed for the case of vanishing RR and NS-NS forms at $k = 0$ (and $k = 1$) the equations of motion (2.8) are simple enough to be solved directly. The equations of motion

$$\lambda'' = 0, \qquad \varphi'' + ce^{-2\varphi} = 0 \tag{4.10}$$

are solved by

$$\lambda' = b\,, \qquad \varphi' = \sqrt{c}\left(a^2 + e^{-2\varphi}\right)^{1/2}. \tag{4.11}$$

Plugging this into the zero-energy condition we find:

$$nb^2 = ca^2\,. \tag{4.12}$$

For $a = b = 0$ the equation for φ is solved by $e^{\varphi} = \sqrt{c}\,\rho$ or $\varphi = \sqrt{c}\,\tau$ so that the string coupling is $e^{2\phi} \sim e^{\sqrt{c}\tau}$. This is obviously the linear dilaton solution associated with the superpotential $w = \sqrt{c}$ which was discussed above.

For $a \neq 0$ the solution of the equations reads

$$\lambda = -\sqrt{\frac{c}{n}}\rho, \qquad e^{\varphi} = \frac{1}{a}\sinh(\sqrt{c}\,a\rho) \tag{4.13}$$

or in terms of the τ coordinate

$$e^{\lambda} = \left[\tanh\left(\frac{1}{2}\sqrt{c}\,\tau\right)\right]^{1/\sqrt{n}} \qquad \text{where} \qquad e^{\sqrt{c}\tau} = \tanh\left(\frac{1}{2}\sqrt{c}\,\rho\right). \tag{4.14}$$

Now let us come back to the second type of superpotential discussed above. Plugging this solution into (4.11) it is easy to verify that indeed equations (4.9) are identical to (4.11). Thus we have established a one to one map between the solutions of the BPS equations and the solutions of the equations of motion. The dilaton derived from this solution is given by:

$$e^{2\phi} = \frac{1}{a}\frac{\left[\tanh\left(\frac{1}{2}\sqrt{c}\,\tau\right)\right]^{\sqrt{n}}}{\sinh\left(\sqrt{c}\,\tau\right)} \tag{4.15}$$

and we see that for large enough a we have a small string coupling at any τ (see figure 1). For $\tau \to \infty$ the solution reduces to the linear dilaton background with the flat minkowskian

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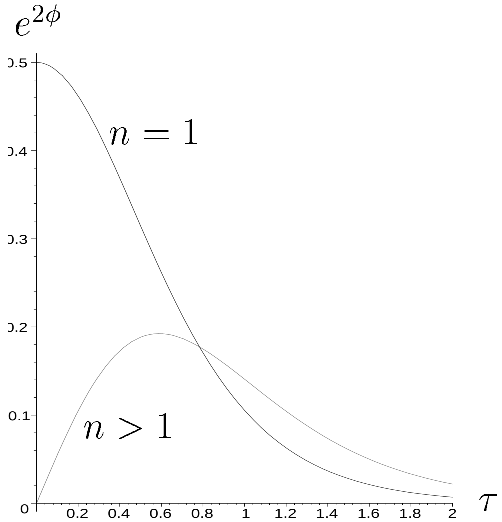

Figure 1: The string coupling vs. the radial coordinate τ for $a = 1$ and $n = 1, 4$. According to (4.15) the maximum value of e^{ϕ} is fixed by a and for large enough a the string coupling is small everywhere.

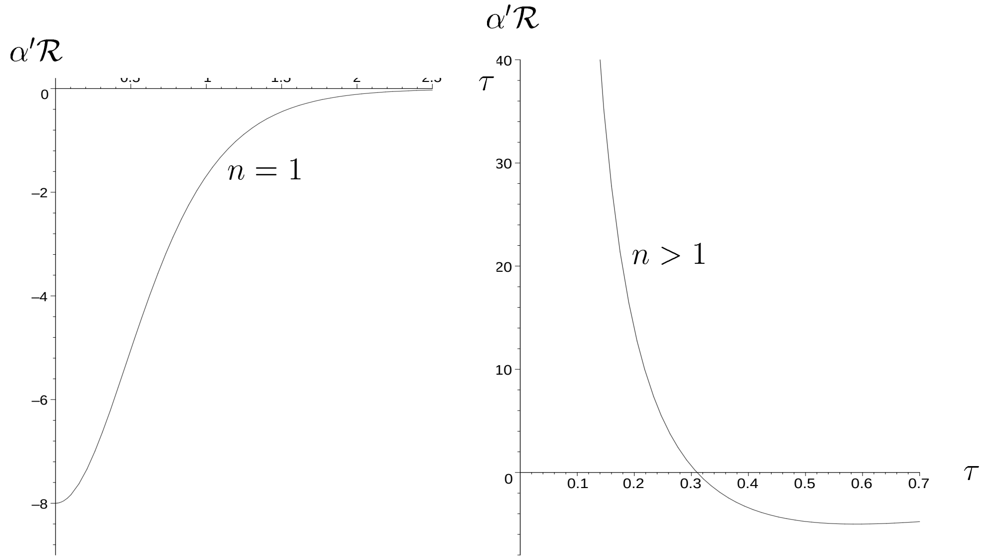

Figure 2: The curvature (4.16) vs. τ. For $n = 1$ there is no singularity, while for $n > 1$ the metric is singular at $\tau \to 0$.

metric discussed above. The scalar curvature associated with this metric is given by

$$l_s^2 \mathcal{R} = -\frac{c(n+1)}{\sinh^2\left(\sqrt{c}\,\tau\right)} \left(\frac{2\sqrt{n}}{n+1} \cosh\left(\sqrt{c}\,\tau\right) - 1 \right). \tag{4.16}$$

For $n = 1$ the metric is regular everywhere and for $n > 1$ there is a naked singularity at $\tau = 0$ (see figure 2).

This solution was originally derived in [28] in the context of bosonic strings. It was further shown that in the linear approximation the closed strings tachyon equation of motion has an exact "kink" solution, which interpolates between different expectation values.

4.2 $k = 1$, the cylinder, the cigar and beyond

In this case the compact transverse space is an S^1. Let's start again with the equations of motion

$$\partial_\rho^2 \lambda = 0\,, \qquad \partial_\rho^2 \nu = 0\,, \qquad \partial_\rho^2 \varphi = -ce^{-2\varphi}\,. \tag{4.17}$$

So that:

$$\partial_\rho \lambda = b_\lambda \rho\,, \qquad \partial_\rho \nu = b_\nu \rho\,, \qquad \partial_\rho \varphi = c^{\frac{1}{2}} \left(a^2 + e^{-2\varphi}\right)^{1/2}. \tag{4.18}$$

The zero energy condition implies $nb_\lambda^2 + b_\nu^2 = ca^2$ so it is convenient to parameterize the solution as follows

$$b_\lambda = \sqrt{\frac{c}{n}} a \cos\beta \qquad \text{and} \qquad b_\nu = \sqrt{c}\, a \sin\beta\,. \tag{4.19}$$

For $a = 0$ λ and ν are constants. Again with no loss of generality we take them to be $\lambda = \nu = 0$ so that we get a geometry of a *cylinder*:

$$ds^2 = -dt^2 + \cdots + dx_{n-1}^2 + d\tau^2 + d\theta^2 \tag{4.20}$$

with a linear dilaton

$$\phi = \frac{1}{2}\sqrt{c}\tau\,. \tag{4.21}$$

As we have already mentioned this solution corresponds to an exact 2d conformal theory on the world-sheet [27] (now one of the coordinates is compact).

A more interesting background is found for $a \neq 0$ where we have $e^{-c^{\frac{1}{2}}\tau} = \tanh\left(\frac{1}{2}\sqrt{c}\, a\rho\right)$ and hence

$$e^\lambda = \left[\coth\left(\frac{1}{2}\sqrt{c}\,\tau\right)\right]^{\frac{\cos\beta}{\sqrt{n}}} \qquad e^\nu = \left[\coth\left(\frac{1}{2}\sqrt{c}\,\tau\right)\right]^{\sin\beta}. \tag{4.22}$$

Substituting this into the expression for the curvature and requiring regularity at $\tau = 0$ we obtain the following equation for β:

$$\sqrt{n}\, \cos\beta + \sin\beta = -1\,, \tag{4.23}$$

which is solved by by two types of solutions. One solution is simply $\sin\beta = -1$. This implies $\lambda = 0$ and $e^\nu = \tanh\left(\frac{1}{2}\sqrt{c}\,\tau\right)$, which is nothing but the famous *cigar* background, or more precisely $R^{1,p} \times$ *cigar*, namely

$$ds^2 = -dt^2 + \cdots + dx_{n-1}^2 + d\tau^2 + \tanh^2\left(\frac{1}{2}\sqrt{c}\,\tau\right) d\theta^2 \tag{4.24}$$

with a dilaton of the form

$$e^{2\phi} = \frac{1}{2a} \frac{1}{\cosh^2\left(\frac{1}{2}\sqrt{c}\,\tau\right)}\,. \tag{4.25}$$

The radius of the compact coordinate in the metric (4.24) is equal to $R_\theta = 2/\sqrt{c}$. It is fixed by requiring the space to be regular at $\tau = 0$.

The scalar curvature of this "cigar" background is (see figure 3):

$$l_s^2 \mathcal{R} = -\frac{c}{\cosh^2\left(\frac{1}{2}\sqrt{c}\,\tau\right)}\,. \tag{4.26}$$

The cigar background like the cylinder one corresponds to an exact string solution [29] (see also [30] a recent discussion of the topic).

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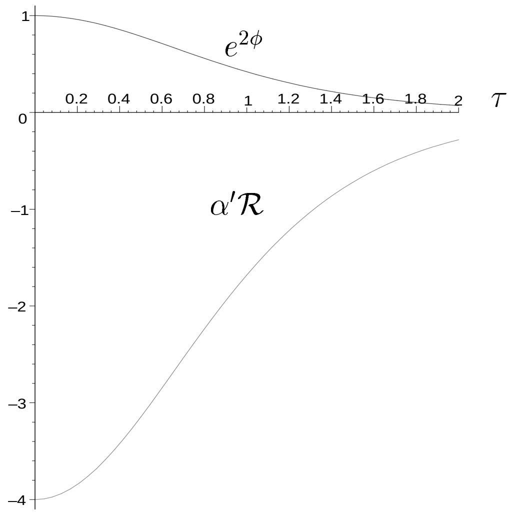

Figure 3: The curvature (4.26) and the dilaton (4.25) of the "cigar" solution for $a = 1$ and $c = 4$.

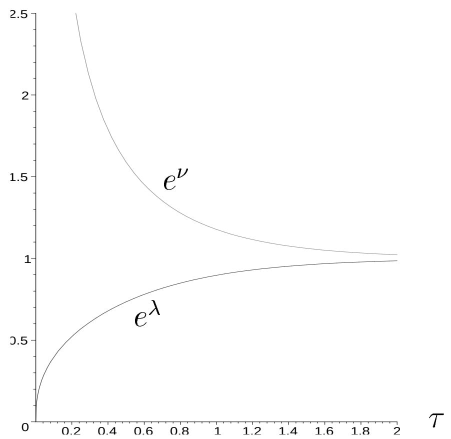

Figure 4: The warp factors e^{λ} and e^{ν} (4.28) vs. τ for $c = 4$.

The other solution of eq. (4.23) is

$$\cos\beta = -\frac{2\sqrt{n}}{n+1}\,, \qquad \sin\beta = \frac{n-1}{n+1}\,. \tag{4.27}$$

The corresponding components of the metric read (see figure 4)

$$e^{\lambda} = \left[\tanh\left(\frac{1}{2}\sqrt{c}\,\tau\right)\right]^{\frac{2}{n+1}} \qquad e^{\nu} = \left[\tanh\left(\frac{1}{2}\sqrt{c}\,\tau\right)\right]^{-\frac{n-1}{n+1}}\,. \tag{4.28}$$

Remarkably, the dilaton and the curvature of this solution are precisely like in the cigar background. The two backgrounds, however, are not related through a coordinate transformation for $n > 0$, as can be seen from the non-trivial warp factor e^{λ}. As a consistency check one may easily verify that for $n = 0$ the solution reduces to the cigar background without the $R_{n-1,1}$ part in the metric. In contrast to the cigar solution the metric given by (4.28) has a horizon at $\tau = 0$. Moreover, this point cannot anymore be thought of as the tip of the cigar since (at least for $n > 1$) the warp factor e^{ν} diverges at $\tau = 0$.

It will be interesting to see whether this configuration with the non-trivial warp factor $e^{\lambda(\tau)}$ might appear as an exact world-sheet solution similar to the cigar background.

Before moving on to higher dimensional transverse spaces, let us briefly remark about the superpotential approach to the $k = 1$ case. A superpotential that corresponds to $V = ce^{-2\varphi}$ for the case of a non-trivial ν is

$$W = -4\sqrt{c}\,e^{-\varphi}\cosh(\nu)\,. \tag{4.29}$$

The corresponding BPS equations are

$$\begin{aligned} \partial_\rho\varphi &= -\frac{1}{4}\partial_\varphi W = -\sqrt{c}\,e^{-\varphi}\cosh(\nu) \\ \partial_\rho\nu &= \frac{1}{4}\partial_\nu W = -\sqrt{c}\,e^{-\varphi}\sinh(\nu)\,. \end{aligned} \tag{4.30}$$

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To solve the corresponding BPS equations it is convenient to rewrite these equations in terms of derivatives with respect to τ. The second equation then reads

$$\partial_\tau \nu = \frac{1}{4} e^{\varphi} \partial_\nu W = -\sqrt{c}\, \sinh(\nu)\,, \tag{4.31}$$

which admits a solution of the form

$$e^{\nu} = \tanh\left(\frac{1}{2}\sqrt{c}\,\tau\right). \tag{4.32}$$

Plugging this back into the first equation in (4.30) one finds the dilaton solution of (4.25). This combines with $e^{2\lambda} = 1$ is obviously the cigar solution. For completeness we present in appendix D the derivation of the cigar solution in the Einstein frame using the formulation of (2.19).

4.3 The cigar and trumpet solutions as T-duals

In the last subsection we have discussed solutions characterized by a compact S_1 transverse space. This naturally calls for the implementation of T-duality to generate new solutions of the equations of motion. In the present context T-duality acts on e^{ν} and on e^{ϕ} as follows

$$e^{2\nu} \;\rightarrow\; e^{-2\nu} \qquad e^{2\phi} \;\rightarrow\; e^{2\phi-2\nu} \tag{4.33}$$

where we still use $\alpha' = 1$. The cylinder solution given in (4.20) is unaltered by T duality. For a constant radius different from 1 the transformation is also obvious.

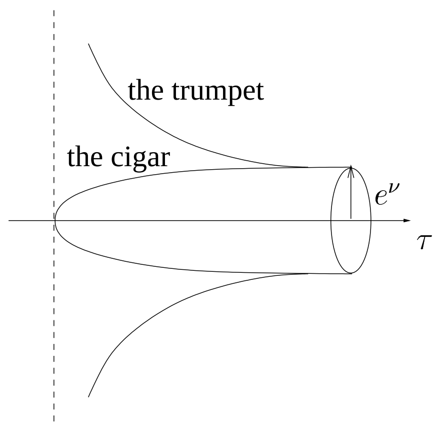

Figure 5: The "cigar" and the "trumpet" geometry. In the former the factor e^{ν} vanishes at $\tau = 0$ (the tip), while in the later $e^{\nu} \to \infty$ and the curvature diverges.

The more interesting cases are of course the cigar solution and its generalizations. Replacing ν with $-\nu$ in (4.31) is clearly a symmetry transformation of this equation and therefore indeed in addition to the cigar solution there is also a *trumpet* solution of the form

$$e^{\nu} = \coth\left(\frac{1}{2}\sqrt{c}\,\tau\right) \qquad e^{2\phi} = \frac{1}{2a}\frac{1}{\sinh^2\left(\frac{1}{2}\sqrt{c}\,\tau\right)} \tag{4.34}$$

Hence the trumpet solution is the T-dual solution of the cigar solution. Both solutions are drawn in figure 5.

4.4 SUGRA background with a four dimensional compact transverse space ($k = 4$)

The main difference between the S_1 compact transverse space and the general S_k case is the curvature contribution to the potential in the latter. Still with no RR or NS form the potential takes the form

$$V = -k(k-1)e^{-2\nu-2\varphi} - ce^{-2\varphi}\,. \tag{4.35}$$

Let us first check whether the cylinder and cigar solutions can be generalized to the higher dimensional cases. For that purpose we assume that the superpotential has the form $W(\varphi,\nu)$ so that the relation between the potential and the superpotential has the following form

$$\frac{1}{k}W_\nu^2 - W_\varphi^2 = 16V\,. \tag{4.36}$$

Again we take an ansatz for the superpotential of the form $W = 4e^{-\varphi}w(\nu)$. which implies that

$$\frac{1}{k}w_\nu^2 - w^2 = -k(k-1)e^{-2\nu} - c\,, \tag{4.37}$$

For $k = 0,1$ solutions were presented in the previous subsection. We were not able to derive a solution for general $k \neq 0,1$ apart from the special case of a four sphere $k = 4$. It is straightforward to verify that for this case the following is a solution for the superpotential

$$w(\nu) = 6\sqrt{c}\,e^{-2\nu} + \sqrt{c}\,. \tag{4.38}$$

As was done above we transform to τ dependence rather than the ρ dependence. In terms of this parameterization the BPS equations are

$$\partial_\tau\nu = \frac{3}{\sqrt{c}}e^{-2\nu}\,, \qquad \partial_\tau\varphi = -\left(6\sqrt{c}e^{-2\nu} + \sqrt{c}\right). \tag{4.39}$$

Finally the solution for ν and φ is

$$e^{2\nu} = 6c^{-1/2}\tau \qquad \text{and} \qquad \varphi = -\ln\tau - \sqrt{c}\,\tau + A\,. \tag{4.40}$$

Now since in this solution λ is a constant which again we take to be $\lambda = 0$ the metric of this non-critical supergravity takes the form

$$ds^2 = dx_{||}^2 + d\tau^2 + 6c^{-1/2}\tau d\Omega_4^2\,, \tag{4.41}$$

Recall that the range of τ is $0 < \tau < \infty$. The corresponding scalar curvature is given by:

$$R = 8\nu'' + 20\nu'^2 - 12e^{-2\nu} = \frac{1-2\sqrt{c}\,\tau}{\tau^2}\,, \tag{4.42}$$

which is singular at $\tau \to 0$.

The dilaton on the other hand is regular everywhere as can be seen from the following expression

$$2\phi = \varphi + 4\nu\,, \tag{4.43}$$

and hence

$$e^{2\phi} \sim \tau e^{-\sqrt{c}\,\tau}\,. \tag{4.44}$$

5. Conformal $AdS_{n+1} \times S^k$ backgrounds

Next we consider non-critical backgrounds that incorporate RR forms. Maldacena's duality in its original form relates the $AdS_5 \times S^5$ critical string (supergravity background) with the of $\mathcal{N} = 4$ SYM conformal field theory that lives on the boundary of the AdS_5 (in a large-N limit). The $AdS_5 \times S^5$ solution was derived as a near horizon limit of the supergravity background of N D3 branes, but obviously one can generate this solution directly with no reference to the near horizon limit. In the search for non-critical supergravity backgrounds that admit gauge theory duals, we will follow the latter approach. Following the steps of the holographic duality for critical strings, we first look for "conformal non-critical backgrounds" namely non-critical backgrounds with a constant dilaton.

A brief glance over (2.9) tells us that requiring a constant dilaton implies also a constant ν

$$\partial_\rho \phi = 0 \quad \rightarrow \quad \partial_\rho \nu = 0 \tag{5.1}$$

and the solution of this condition takes the form

$$\begin{aligned} e^{2\phi_0} &= \frac{1}{n+1-k}\left(\frac{(n+1-k)(k-1)}{c}\right)^k \frac{2c}{Q^2} \\ e^{2\nu_0} &= \frac{(n+1-k)(k-1)}{c}\,. \end{aligned} \tag{5.2}$$

In order not to have infinite string coupling or vanishing warp factor of the world-volume coordinates, we must require

$$n+1-k \neq 0 \qquad k \neq 1\,. \tag{5.3}$$

It is convenient at this stage to switch from ρ to τ dependence. Recalling that $\varphi = 2\phi - n\lambda - k\nu$ we find that the equation for λ is

$$\partial_\tau^2 \lambda + n(\partial_\tau \lambda)^2 = \frac{Q^2}{2} e^{2\phi_0 - 2k\nu_0}\,. \tag{5.4}$$

This is solved by

$$\lambda = \left(\frac{c}{n(n+1-k)}\right)^{1/2} \tau + \lambda_0\,. \tag{5.5}$$

It is easy to check that this solution is in accordance with the zero energy condition.

Defining $R_{AdS}^{1/2} u = e^{\tau R_{AdS}^{-1/2}}$ we end up with the following metric:

$$l_2^{-2} ds^2 = ds^2_{AdS_{n+1}} + ds^2_{S^k} = \left(\frac{u}{R_{AdS}}\right)^2 dx_\parallel^2 + \left(\frac{R_{AdS}}{u}\right)^2 du^2 + R_{S^k}^2 d\Omega_k^2\,, \tag{5.6}$$

where

$$R_{AdS} = \left(\frac{n(n+1-k)}{c}\right)^{1/2} \qquad \text{and} \qquad R_{S^k} = \left(\frac{(n+1-k)(k-1)}{c}\right)^{1/2}. \tag{5.7}$$

The implications of this result is that for any d dimensional space-time there are several solutions of the form $AdS_{n+1} \times S^k$ such that $n+1 \neq k$ and $k \neq 1$. For instance for $d=6$ the space-time may be of the following types:

$$AdS_6\,, \qquad AdS_4 \times S^2\,, \qquad AdS_2 \times S^4 \tag{5.8}$$

or from a different point of view, four dimensional boundary space-time associated with AdS_5 can be accompanied by S^3 in addition to the well known solution with S^5 if we restrict ourself to even d (as will be required in section 9) or also S^0, S^2 and S^4 for general d.

Anticipating the discussion of the dual gauge theory (see section 9) it is remarkable that $e^\phi N$, which will map into 't Hooft coupling constant of the dual gauge theory, appears to be Q-independent:

$$g_s N = e^\phi N = \left(\frac{2c}{n+1-k}\left(\frac{(n+1-k)(k-1)}{c}\right)^k\right)^{1/2}\,, \tag{5.9}$$

where we have ignored the numerical factors in the relation between Q and N.

5.1 The $AdS_n \times S^k$ backgrounds from BPS equations

Let us now prove that the solution presented above is supersymmetric, namely, that it can be derived from a superpotential. (Strictly speaking, being a solution of the BPS equation is not a sufficient condition for a supersymmetric solution but it clearly is a necessary condition). If there is a superpotential that generates the potential, then the profiles of ϕ and ν are determined by the BPS equations. Since we require a constant dilaton, which as we saw implies a constant ν, we have to show that at $\phi = \phi_0$ and $\nu = \nu_0$

$$\partial_\phi W|_{\phi=\phi_0,\nu=\nu_0} = 0 \qquad \partial_\nu W|_{\phi=\phi_0,\nu=\nu_0} = 0\,. \tag{5.10}$$

If we expand the superpotential around ϕ_0 and ν_0 this means that there should not be any liner terms in ϕ and ν.

It is convenient to introduce new variables x and y:

$$x = \frac{1}{n-1}(\lambda + \varphi) \qquad \text{and} \qquad y = \frac{1}{n-1}(n\lambda + \varphi) = \frac{1}{n-1}(2\phi - k\nu)\,. \tag{5.11}$$

Note that y is a constant for the case of our interest since both ϕ and ν are. In terms of ν, x and y the superpotential equation reads:

$$-\frac{1}{n(n-1)}W_x^2 + \frac{1}{n-1}W_y^2 + \frac{1}{k}W_\nu^2 = 16e^{-2nx}\left(Q^2 e^{(n+1)y-k\nu} - ce^{2y} - k(k-1)e^{2(y-\nu)}\right). \tag{5.12}$$

One may simplify this equation substituting $W = -4e^{-nx}w(y,\nu)$:

$$-\frac{n}{n-1}w^2 + \frac{1}{n-1}w_y^2 + \frac{1}{k}w_\nu^2 = Q^2 e^{(n+1)y-k\nu} - ce^{2y} - k(k-1)e^{2(y-\nu)}\,. \tag{5.13}$$

It is remarkable that expanding the expression on the r.h.s. of (5.13) around the point $y = y_0 = \frac{1}{n-1}(2\phi_0 - k\nu_0)$ and $\nu = \nu_0$, where ϕ_0 and ν_0 are given in (5.2), we do not obtain any term linear in $(y-y_0)$ or $(\nu-\nu_0)$:

$$Q^2 e^{(n+1)y_0-k\nu_0}\Big(-\frac{1}{2}(n-1)+\frac{1}{2}(n^2-1)(y-y_0)^2- \qquad (5.14)$$
$$-k(n+3)(y-y_0)(\nu-\nu_0)+\frac{1}{2}(k-2)(\nu-\nu_0)^2\Big)+\text{higher order terms}\,.$$

It immediately follows that the function $w(y,\nu)$ has a similar form, namely:

$$w(y,\nu) = \frac{n-1}{(2n)^{1/2}}Qe^{\frac{1}{2}((n+1)y_0-k\nu_0)}+$$
$$+\left(\frac{1}{2}A(y-y_0)^2+B(y-y_0)(\nu-\nu_0)+\frac{1}{2}C(\nu-\nu_0)^2\right)+\cdots \qquad (5.15)$$

for some constant A, B and C. Plugging this result into the equation of motion of λ, ν and φ we find the solution given in the beginning of this section. Explicitly, $\dot\nu=\dot\phi=0$ since W_ν and W_ϕ vanish at $\nu=\nu_0$ and $\phi=\phi_0$, while for λ we get $\dot\lambda=R^{-1}_{AdS}$ in agreement with the previous result.

5.2 AdS black hole solutions

In is well known that on top of extremal SUGRA backgrounds (in critical dimensions), one can construct near extremal solutions which correspond to boundary field theories at finite temperature. In particular such backgrounds were written down for the near extremal Dp branes [31]. For $D3$ brane in the near horizon limit the near extremal solution is the AdS black hole solution. Since we have identified a family of $AdS_{n+1}\times S^k$ backgrounds we would like to examine now whether one can turn them into near extremal solutions, namely, determine non-critical AdS black hole solutions.

For this purpose we consider now the following generalization of our initial ansatz:

$$l_s^{-2}ds^2=-e^{2\tilde\lambda(\tau)}dt^2+\sum_{i=1}^{n-1}e^{2\lambda(\tau)}dx_i^2+d\tau^2+e^{2\nu(\tau)}d\Omega_k^2\,. \qquad (5.16)$$

Obviously such solutions will be non-supersymmetric and hence there is no sense to explore them via the BPS equations but rather by the 2nd order equations of motion. We look for solutions with constant ν and ϕ. We first solve the equations for ν and ϕ exactly as in the supersymmetric $\lambda=\tilde\lambda$ case and then we arrive at the following equations for λ and $\tilde\lambda$:

$$\partial_\rho^2\tilde\lambda=\frac{1}{2}Q^2e^{2(\tilde\lambda+(n-1)\lambda-\phi_0)} \qquad \text{and} \qquad \partial_\rho^2\lambda=\frac{1}{2}Q^2e^{2(\tilde\lambda+(n-1)\lambda-\phi_0)}\,. \qquad (5.17)$$

The most general solution of this system is:

$$\tilde\lambda=-\frac{1}{n}\ln\left(\frac{1}{a}\sinh(a\alpha\rho)\right)+(n-1)b\rho \qquad \text{and} \qquad \lambda=-\frac{1}{n}\ln\left(\frac{1}{a}\sinh(a\alpha\rho)\right)-b\rho \quad (5.18)$$

where $\alpha^2=\frac{1}{2}nQ^2e^{-2\phi_0}$ and a and b related through the zero energy condition:

$$b=-\frac{\alpha}{n}a\,. \qquad (5.19)$$

In order to rewrite the metric in a more familiar form we will use a new radial coordinate u defined by:

$$e^{\lambda} = \frac{u}{R_{AdS}} \,. \tag{5.20}$$

In terms of this coordinate the metric might be easily re-written as:

$$l_s^{-2} ds^2 = \left(\frac{u}{R_{AdS}}\right)^2 \left[-\left(1 - \left(\frac{u_0}{u}\right)^n\right) dt^2 + dx_i^2 \right] + \left(\frac{R_{AdS}}{u}\right)^2 \frac{du^2}{\left(1 - \left(\frac{u_0}{u}\right)^n\right)} + R_{S^k}^2 d\Omega_k^2 \,, \tag{5.21}$$

where the energy density on the brane is given by $u_0^n = 2aR_{AdS}^n$. Obviously for zero energy density ($a = 0$) we are back in the extremal supersymmetric solution. The holographic interpretation of these near extremal solutions will be addressed in section 9.

6. The RR deformed two dimensional black hole

The next goal of our program is to look for non-critical, non conformal backgrounds, namely with non-constant dilaton, with non-trivial RR forms. We start this journey with a two dimensional non-critical SUGRA background.

Recall (2.6) that the contribution of the RR charge to the potential has the form $Q^2 e^{n\lambda - k\nu - \varphi}$. In the case of no transverse compact space $k = 0$ this is just the potential of a cosmological constant Λ with $\Lambda = -Q^2 > 0$. Altogether the potential now reads

$$V = Q^2 e^{n\lambda - \varphi} - ce^{-2\varphi} \,. \tag{6.1}$$

The superpotential is defined via equation (2.17) which now is $\frac{1}{n} W_\lambda^2 - W_\varphi^2 = 16V$. As we have done previously we now factor out $e^{-\varphi}$ so that $W = 4e^{-\varphi} w(\phi)$. We will see later that in general it is convenient to take $W = 4e^{-\varphi} w(z)$ where $z = n\lambda - k\nu + \varphi$ which in our case reduces to $z = n\lambda + \varphi = 2\phi$. For $n = 1$ the equation for $w(\phi)$ is given by

$$w'(\phi) w(\phi) - w(\phi)^2 = Q^2 e^{2\phi} - c \,. \tag{6.2}$$

This equation has an analytic solution [24]:

$$w(\phi) = \sqrt{2\phi Q^2 e^{2\phi} - 4me^{2\phi} + c} \,, \tag{6.3}$$

where m is an integration constant. From the BPS equations we have:

$$\frac{\partial\lambda}{\partial\tau} = \frac{1}{2} w'(\phi) \qquad \frac{\partial\varphi}{\partial\tau} = w(\phi) - \frac{1}{2} w'(\phi) \,, \tag{6.4}$$

which implies in particular that:

$$\frac{\partial\phi}{\partial\tau} = \frac{1}{2} w(\phi) \,. \tag{6.5}$$

It turns out that it is useful to take ϕ as a radial coordinate instead of τ. It is easy to check that in this parameterization $e^{2\lambda} = w^2$ so that we we end up with the 2d black hole metric of [24]

$$l_s^{-2} ds^2 = -\frac{1}{4} w^2(\phi) dt^2 + \frac{d\phi^2}{\frac{1}{4} w^2(\phi)} \,, \tag{6.6}$$

In [24] the metric is expressed in terms of $l(\phi) = \frac{1}{4}w^2(\phi)$. It was shown in [24] that this solution can be interpreted as a two dimensional black hole with an ADM mass $M_{ADM} = \frac{2}{\sqrt{c}}m$. The scalar curvature was found to be $\alpha'\mathcal{R} = e^{2\phi}[Q^2(\phi+1) - m]$.

In fact using ϕ as the radial coordinate, one can derive this solution directly from the equations of motions. In appendix E we find that the function $e^{2\lambda} = l(\phi)$ is indeed the same as derived from the BPS equations. The points $\phi = \phi_0$ where $w(\phi) = 0$ constitute the horizon. In the extremal case $l(\phi)$ has a double zero at $\phi = \phi_0$ and the space-time geodesics are complete (the integral $\int \frac{d\phi}{l^{\frac{1}{2}}(\phi)}$ diverges at ϕ_0). The double zero requirement leads to

$$e^{2\phi_0} = \frac{c}{Q^2} \qquad \text{and} \qquad m = \frac{1}{4}Q^2(1+2\phi_0)\,. \tag{6.7}$$

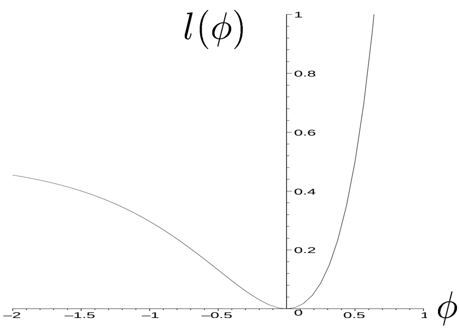

Figure 6: The function $l = l(\phi)$ for $Q^2 = 2$, $m = \frac{1}{2}$ and $\phi = \phi_0 = 0$. Note that $l(\phi)' = l(\phi) = 0$ at $\phi_0 = 0$ in agreement with (6.7).

The function $l(\phi)$, which satisfies the above conditions is plotted in figure 6.

In [24] the near horizon limit of the black hole solution was shown to be a two dimensional AdS solution. Expanding $l(\phi)$ around ϕ_0 takes the form $l(\phi) \sim \frac{c}{2}(\phi - \phi_0)^2$ since $l(\phi_0) = l'(\phi_0) = 0$. Defining now a new variable $u = \phi - \phi_0$ the near horizon metric is

$$l_s^{-2} ds^2 = -\left(\frac{u}{R_{AdS}}\right)^2 dt^2 + \left(\frac{R_{AdS}}{u}\right)^2 du^2\,, \tag{6.8}$$

where $R_{AdS} = \sqrt{2/c}$ and the scalar curvature is given by $\alpha'\mathcal{R}_{AdS} = c$. It will be useful for later purposes to find this metric directly from the second order equations of motion.

$$\partial_\rho^2 \lambda = \frac{1}{2}Q^2 e^{\lambda-\varphi}\,, \qquad \partial_\rho^2\varphi = -ce^{-2\varphi} + \frac{1}{2}Q^2 e^{\lambda-\varphi}\,, \tag{6.9}$$

thus

$$\partial_\rho^2(\lambda+\varphi) = (-c + Q^2 e^{\lambda+\varphi})e^{-2\varphi}\,. \tag{6.10}$$

This equation is solved by (6.7) in accord with the previous discussion. Plugging this result into the equation for φ we get:

$$\partial_\rho^2\varphi = -\frac{c}{2}e^{-2\varphi}\,. \tag{6.11}$$

This is easily solved by $e^\varphi = \sqrt{\frac{c}{2}}\,\rho$ and therefore $\varphi = \sqrt{\frac{c}{2}}\,\tau$. Using the result for the dilaton we find that:

$$e^\lambda = \sqrt{\frac{c}{2}}\,e^{\sqrt{\frac{c}{2}}\tau}\,. \tag{6.12}$$

Defining $u = e^{\sqrt{\frac{c}{2}}\tau}$ we arrive at the AdS metric.

Few remarks are in order:

- In fact this type of solution with constant dilaton exists also for general n. It is easy to verify that the corresponding solution is

$$e^{2\phi_0} = \frac{2}{n+1}\frac{c}{Q^2} \qquad \lambda = \left(c\frac{n}{n+1}\right)^{\frac{1}{2}}\tau + \lambda_0\,. \tag{6.13}$$

  Defining $u = e^{\left(c\frac{n}{n+1}\right)^{\frac{1}{2}}\tau}$ we arrive at the following AdS metric:[4]

$$l_s^{-2}ds^2 = \left(\frac{u}{R_{AdS}}\right)^2 dx_\mu^2 + \left(\frac{R_{AdS}}{u}\right)^2 du^2\,, \tag{6.14}$$

  where $R_{AdS} = \sqrt{\frac{n(n+1)}{c}}$.

- Note that since the string coupling is $g_s = e^{\phi_0} = \frac{2}{n+1}\frac{\sqrt{c}}{Q}$, large RR charge Q means weak string coupling so that we can trust the perturbative string description.

- On the other hand the since the scalar curvature is $R = \frac{c}{n(n+1)}$ it does not depend on the number of branes and remains fixed in the large N limit. This is obviously not like for the $AdS_5 \times S_5$ case, where the small curvature requires large N. In fact this is a particular AdS solution discussed in section 5.

6.1 The T-dual solution of the BGV black hole $k = 1$

Consider now instead of a space-time with one world volume coordinate and no compact transverse space ($k = 0, n = 1$) the two dimensional case with no world volume coordinate but with an S^1 direction namely ($k = 1, n = 0$). Recall that the potential for $k = 1$ takes the form

$$V = Q^2 e^{-\nu-\varphi} - ce^{-2\varphi}\,. \tag{6.15}$$

Using again the following parameterization of the superpotential

$$W = 4e^{-\varphi}w(z)\,, \qquad \text{where} \qquad z = -\nu + \varphi\,. \tag{6.16}$$

The superpotential equation reads

$$2w'(z)w(z) - w(z)^2 = Q^2e^z - c \tag{6.17}$$

and therefore

$$w^2(z) = Q^2ze^z - Ae^z + c\,, \tag{6.18}$$

where A is constant. It means that the metric is

$$l_s^{-2}ds^2 = \frac{1}{w^2(z)}\left(dz^2 + d\theta^2\right), \tag{6.19}$$

[4] For $n = 1$ this result appears in BGV.

where $d\tau = \frac{dz}{w(z)}$ and the dilaton is given by $e^{2\phi} \sim w^{-2}(z)$. In fact it is easy to realize that this $(k = 1, n = 0)$ solution is just the T-dual of the BGV solution. As discussed for backgrounds with no RR charges T-duality implies inverting the radius of the compact direction and transforming the dilaton $e^{\phi} \to \frac{e^{\phi}}{g_{\text{compact}}}$ where g_{compact} is the metric along the compact direction. For the cigar/trumpet duality these transformation were $\nu \to -\nu$ and $e^{2\phi} \to e^{2\phi-2\nu}$. In the present case the situation is slightly more complicated due to the RR flux. The BGV background is equipped with a RR zero form with one dimensional world volume. Taking the latter to be compact and performing T duality along this circle transforms the RR flux into a 1-form RR flux with no world volume but with additional transverse compact one dimensional transverse space. This is indeed the passage from $(k = 0, n = 1)$ to $(k = 1, n = 0)$. Now the original world volume compact coordinate with metric $e^{2\lambda}$ is transformed into a compact transverse direction with metric $e^{2\nu} = e^{-2\lambda}$ and g_{compact} for the dilaton transformation is $g_{\text{compact}} = e^{2\lambda}$. Thus T-duality is summarized in the following transformations[5]

$$(k = 0, n = 1) \;\to\; (k = 1, n = 0) \qquad \lambda \;\to\; -\nu \qquad e^{2\phi} \;\to\; e^{2\phi-2\nu}\,. \tag{6.20}$$

Similarly to the the BGV solution the function $w(z)$ in (6.18) depends on the integration constant A. In (6.6) this constant was fixed by requiring that the metric should be geodesically complete near the horizon defined by $w(z_0) = 0$. After T-duality, however, the horizon is removed to $z = \infty$ and at the point $w(z_0) = 0$ one has instead a diverging string coupling. Therefore we would like to require that $w(z) \neq 0$ for any z. It leads to (see figure 7 for $w^{-2}(z)$ vs. z for different values of A):

$$A \leq A_{\text{crit}} = Q^2\left(1 + \ln\frac{c}{Q^2}\right). \tag{6.21}$$

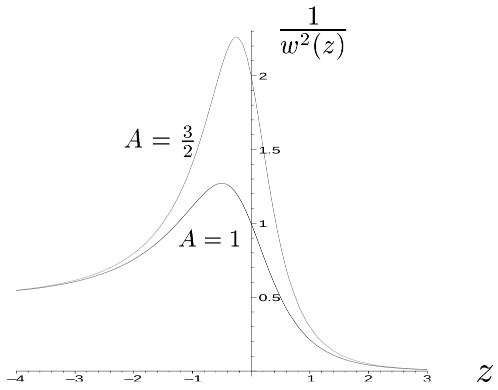

Figure 7: The function $w^{-2}(z)$ for $Q^2 = 2$ and $A = 1, \frac{3}{2}$.

For $A < A_{\text{crit}}$ the solution interpolates between the cylinder at $z \to -\infty$ and a horizon at $z \to \infty$, where $w^{-2}(z) \approx 0$. The curvature of the solution is given by $\alpha'\mathcal{R} = 2(w'^2 - w''w)$ and it diverges for large z. The string coupling $e^{\phi} \sim \frac{1}{w(z)}$, however, is finite everywhere getting its maximal value at the point $z = z_0$ satisfying $w'(z) = 0$.

It is worth to remark that if the above inequality is precisely satisfied ($A = A_{\text{crit}}$), then near the point $w(z_0) = 0$ the metric (6.19) reads:

$$l_s^{-2}ds^2 = \left(\frac{R_{AdS}}{u}\right)^2 (du^2 + d\theta^2) \tag{6.22}$$

and the dilaton behaves like $e^{2\phi} \sim u^{-2}$. To derive this result we have applied the T-duality rules on the AdS metric (6.8). Remarkably the new metric again describes AdS_2 (with a non-constant dilaton and euclidean signature). Unfortunately, at $u \to 0$ the string coupling diverges making the supergravity description irrelevant.

[5]Note that the we also modify the signature of the metric.

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7. Backgrounds with non-zero RR charge $Q \neq 0$ that asymptote to the linear dilaton solution

In section section 4.2 the non-critical cigar and trumpet backgrounds were determined for cases with no RR flux. Let us now examine the fate of these backgrounds once we turn on a RR flux. As we will see in section 9 these SUGRA backgrounds will turn out to be of special interest in the search of duals of the $\mathcal{N}=1$ SYM gauge theories. To study the deformation of the cigar solution, we have to investigate the superpotential equation (2.17) with $k=1$, $Q \neq 0$ and arbitrary n. Recall our parameterization of the superpotential $W = 4e^{-\varphi}w(z)$ where $z = n\lambda - \nu + \varphi$ such that

$$nw'(z)^2 + 2w'(z)w(z) - w^2(z) = Q^2 e^z - c\,. \tag{7.1}$$

In this parameterization the BPS equations read

$$\frac{\partial\lambda}{\partial\tau} = w'(z)\,, \qquad \frac{\partial\nu}{\partial\tau} = -w'(z)\,, \qquad \frac{\partial\varphi}{\partial\tau} = -w'(z) + w(z)\,. \tag{7.2}$$

Using these identities we can write the equation for z:

$$\frac{\partial z}{\partial\tau} = nw'(z) + w(z)\,. \tag{7.3}$$

We will now use z as the radial coordinate so that the general form of the metric is:

$$l_s^{-2}ds^2 = e^{2\lambda(z)}dx_\mu^2 + \frac{dz^2}{(nw'(z)+w(z))^2} + e^{2\nu(z)}d\theta^2\,. \tag{7.4}$$

The non-linear equation (7.1) has analytic solutions only for $n=0$ (see the discussion following (6.17)) and $n=-1$, with the latter being unphysical. Since we do not have an analytic solution let us first discuss the asymptotic behavior of the background. For the latter to correspond to a cylinder metric and a linear dilaton, the superpotential should asymptote as follows $W \to \pm\sqrt{c}\,e^{-\varphi}$, namely, $w(z) \to \pm\sqrt{c}$. Taking $z \to -\infty$ as the asymptotic region, (7.1) implies that asymptotically $w(z)' \to 0$. We will choose the "$-$" solution for a later convenience.[6]

The perturbative expansion of $w(z)$ around $z \to -\infty$ is given by:

$$w(z) \approx \sqrt{c} - \frac{1}{2}\frac{Q}{\sqrt{c}}ze^z \qquad \text{for} \qquad z \to -\infty \tag{7.6}$$

and we can calculate an approximate solution for the metric and the dilaton:

$$\lambda(\tau) \approx -\frac{1}{2}\frac{Q}{\sqrt{c}}\tau e^{-\sqrt{c}\,\tau}\,, \qquad \nu(\tau) \approx \frac{1}{2}\frac{Q}{\sqrt{c}}\tau e^{-\sqrt{c}\,\tau}\,, \qquad 2\phi \approx \sqrt{c}\,\tau\,. \tag{7.7}$$

[6]The equation (7.1) is a second degree differential equation and generally there are two families of solution corresponding to the two roots of the second degree equation:

$$w' = -\frac{1}{n}\left(w \pm \sqrt{(n+1)w^2 + n(Q^2e^z - c)}\right)\,. \tag{7.5}$$

The boundary conditions $w(z)|_{z\to\infty} \to \sqrt{c}$ and $w'(z)|_{z\to\infty} \to 0$ are related to the "$-$" choice in (7.5).

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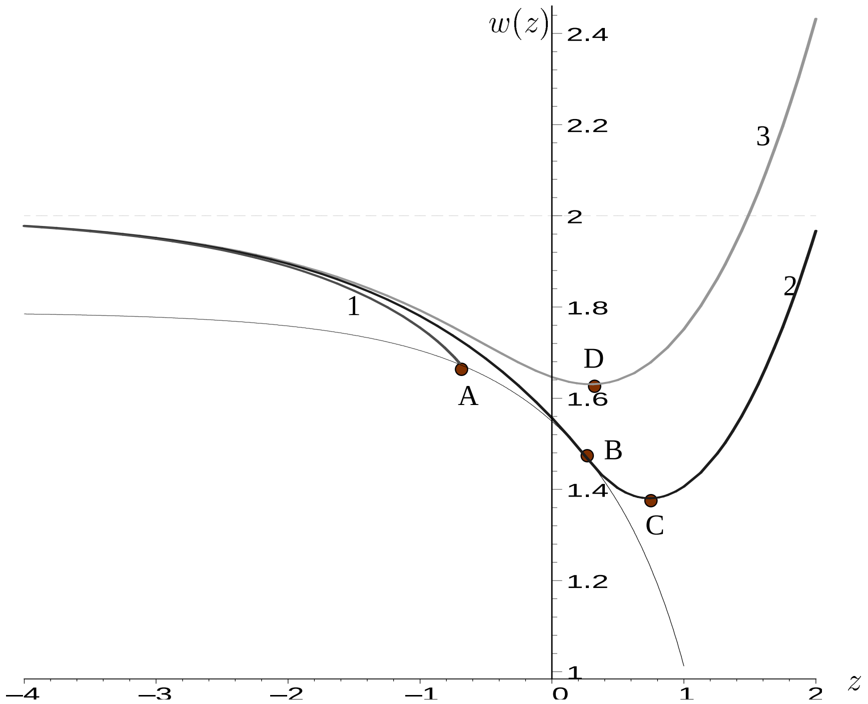

Figure 8: The plot represents three possible solutions (solid curves) of the equation (7.1) for $n = 4$, $Q = 1$. All the solutions approach the asymptotic value $\sqrt{c}$ (the dashed line) at $z \to -\infty$. The numerical expansion related to the curve no. 1 terminates at point A, where it meets the thin curve defined by the (7.8) or alternatively by $nw' + w = 0$. The solution no. 2, however, is only tangent to the $nw' + w = 0$ curve at point B and therefore can be extended to any value of z. Finally, the solution no. 3 does not meet the thin curve, which means that along this curve one always has $nw' + w > 0$. Both no. 2 and no. 3 have $w' = 0$ exactly at one point (C and D respectively).

As expected the warp functions become constant and the dilaton depends linearly on τ. These expressions also show that the region $z \to -\infty$ corresponds to $\tau \to -\infty$. The function $w(z)$ can be extended numerically to the region with finite z. As it shown on figure 8 there are three possible scenario at finite z:

1. This expansion terminates (point A on figure 8), when we approach the curve described by

$$w(z) = \left(\frac{n}{n+1}(c - Q^2 e^z)\right)^{1/2}, \tag{7.8}$$

   which can be also defined by $nw'(z) + w(z) = 0$. Indeed, this is a simple exercise to verify that (7.1) has no solution for $w^2(z) < \frac{n}{n+1}(c - Q^2 e^z)$.

2. The curve that describes the numerical solution (no. 2 on figure 8) is only tangent at $z = z_0$ (point B on figure 8) to the curve defined by (7.8). In this case the numerical solution might be extended to any value of z. In particular, at $z \to \infty$ one has $w(z) \approx \frac{2}{\sqrt{n}} Q e^{z/2}$.

3. In this case the curve (no. 3 on figure 8) does not approach the curve (7.8) and the numerical expansion exists for any z. It is important to no note, that $nw'(z)+w(z) > 0$ along this solution.

Note that from $nw'(z) + w(z) > 0$ follows that $z(\tau)$ is a monotonic function of τ. The completeness of space-time geodesics at infinity (namely at $z = z_0$ where $nw'(z)+w(z) = 0$) implies that the function $nw'(z) + w(z)$ has a simple zero at $z = z_0$. After some algebra we find from this condition that:

$$e^{z_0} = \frac{2}{n+2}\frac{c}{Q^2}, \quad \text{so that} \quad w(z_0) = -nw'(z_0) = \frac{n}{n+1}\alpha, \quad \text{where} \quad \alpha^2 = \frac{n+1}{n+2}c\,. \tag{7.9}$$

The solution no. 1 listed above terminates at $z_A < z_0$ and therefore the space described by this curve is geodesically incomplete. Furthermore, this is a straightforward exercise to verify that at $z = z_0$ the solution of (7.1) is tangent to the curve (7.8) and as a consequence the function $nw'(z)+w(z)$ has a simple zero at this point. This is the curve no. 2 on figure 8 and point B is located precisely at $z = z_0$. Let us describe this solution in more details. The curvature of the metric (7.4) can be written in terms of the function $w(z)$:

$$\alpha'\mathcal{R} = 2(n-1)w''(nw' + w) + (n^2 - n + 2){w'}^2\,, \tag{7.10}$$

where we made use of (7.2). At $z \to -\infty$ (and $\tau \to -\infty$) the solution reduces to the cigar geometry ($\lambda = \nu = 0$) and the curvature goes to zero as expected. At $z \to z_0$ (alternatively $\tau \to \infty$) the curvature has a non-zero value:

$$\alpha'\mathcal{R} = \frac{n^2 - n + 2}{(n+1)(n+2)}c\,. \tag{7.11}$$

In this region we can find the solution using (7.9):

$$\lambda = \frac{\alpha}{n+1}\tau\,, \qquad \nu = -\frac{\alpha}{n+1}\tau\,, \qquad \varphi = -\alpha\tau + \frac{z_0}{2} \qquad \text{and} \qquad \phi = -\frac{\alpha}{n+1}\tau + \frac{z_0}{2} \tag{7.12}$$

and therefore there is a horizon at $\tau \to -\infty$. Defining

$$u = e^{\frac{\alpha}{n+1}\tau} \tag{7.13}$$

we end up with the following background:

$$l_s^{-2}ds^2 = \left(\frac{u}{R_{AdS}}\right)^2 dx_\parallel^2 + \left(\frac{R_{AdS}}{u}\right)^2 du^2 + \left(\frac{R_{AdS}}{u}\right)^2 d\theta^2, \qquad \text{where } R_{AdS} = \frac{n+1}{\alpha} \tag{7.14}$$

and the dilaton is:

$$e^{\phi} = \left(\frac{2}{n+2}\frac{c}{Q^2}\right)^{1/2}\frac{1}{u}\,. \tag{7.15}$$

Actually, the background in the near horizon region is the T-dual version of the AdS solution (6.14) for arbitrary n and $k = 0$. As an additional non-trivial check one can calculate the curvature of the near horizon metric, which should coincide with the previously found result (7.11). To summarize, the background related to the curve no. 2 on figure 8 interpolates between the cylinder solution with a linear dilaton at $\tau \to -\infty$ ($z \to -\infty$) and a configuration consisting of the metric (7.14) and the dilaton (7.15) at $\tau \to +\infty$ ($z \to z_0$). Unfortunately, the dilaton diverges at $u \to 0$ making the supergravity description irrelevant. We will return to this point latter discussing the holographic properties of this solution. In contrast with the cigar solution the warp function of the angular part of the metric diverges at $\tau \to -\infty$, while the world volume warp factor goes to zero (see figure 9). Notice also that the boundary of the metric (7.14) at $u \to \infty$ is four dimensional exactly like in the $AdS_5 \times S_5$ case.

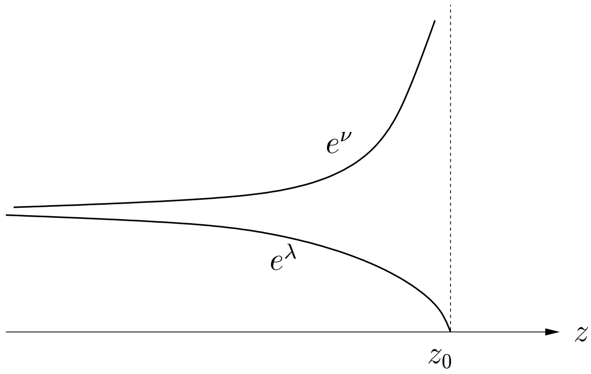

Figure 9: The factors e^{λ} and e^{ν} vs. the coordinate z for the solution corresponding to the curve no. 2 on figure 8. The point $z = z_0$ is mapped to $\tau = -\infty$.

Let us comment briefly on how one arrives at the near horizon solution directly from the second order equations of motion. From (2.8) we get for $k = 1$:

$$\partial_\rho^2(n\lambda - \nu + \varphi) = \left(-c + \frac{1}{2}(n+2)Q^2 e^{n\lambda-\nu+\varphi}\right) e^{-2\varphi} . \tag{7.16}$$

This is trivially solved provided that $z = n\lambda - \nu + \varphi$ is constant satisfying (7.9). Furthermore, from the the equation for φ we obtain:

$$\partial_\rho^2 \varphi = -\alpha^2 e^{-2\varphi} , \tag{7.17}$$

which is solved by $e^{\varphi} = \alpha\rho$. Plugging this into the equations for λ and ν one arrives at the same near horizon background we have found above.

It turns out that there is an additional solution of (7.17), which does not appear as a near horizon limit of the solution with non-constant z:

$$e^{\varphi} = \frac{1}{a}\sinh(\alpha a \rho) . \tag{7.18}$$

Following the same steps as in the $k = 0$ we find

$$\begin{aligned} e^{\lambda} &= \left(4\sinh^2\left(\frac{1}{2}\alpha\tau\right)\right)^{\frac{1}{n+1}} \\ e^{\nu} &= \left(2\sinh\left(\alpha\tau\right)\right)^{-\frac{1}{n+1}} \left(\tanh\left(\frac{1}{2}\alpha\tau\right)\right)^{\frac{n}{n+1}} \\ e^{\phi} &= \left(\frac{2}{n+2}\frac{c}{Q^2}\right)^{1/2} \left(2\sinh\left(\alpha\tau\right)\right)^{-\frac{1}{n+1}} \left(\tanh\left(\frac{1}{2}\alpha\tau\right)\right)^{\frac{n}{n+1}} . \end{aligned} \tag{7.19}$$

For large τ the background degenerates to the near-horizon solution we have identified above. At $\tau = 0$ for $n > 1$ the curvature diverges and there is a naked singularity at this point.

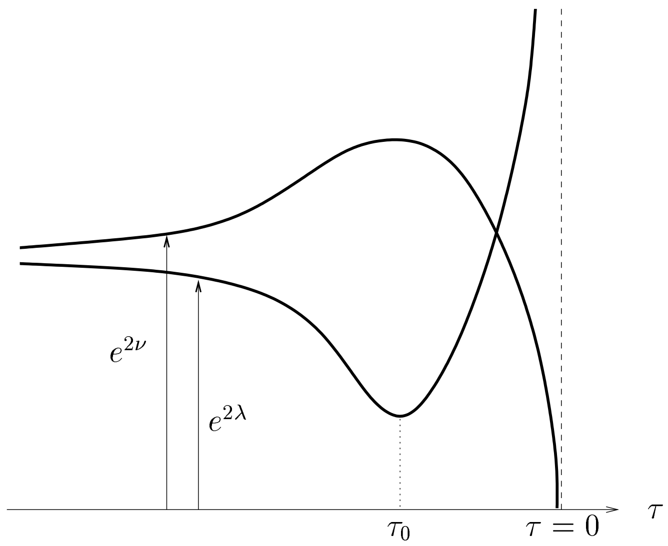

Figure 10: The picture represents the typical form of $g_{ii} = e^{2\lambda}$ and $g_{\theta\theta} = e^{2\nu}$. For $\tau \to -\infty$ we approach the cylinder geometry, while at $\tau \to 0$ the background becomes singular. The function $e^{2\lambda}$ has a global minimum at $\tau = \tau_0$.

Next we will investigate the solution related to the curve no. 3. In this case the function $nw' + w$ does not vanish at any z. Similarly to the curve no. 2 at $\tau, z \to -\infty$ we have $w \approx \sqrt{c}$ and the solution in this region is the cylinder. On the contrary, at $z \to \infty$ the solution of (7.1) is $w \approx \frac{2}{\sqrt{n}} Q e^{z/2}$, so that $\tau \to 0$ and

$$e^{\lambda} \approx \tau^{-\frac{1}{n+2}}, \qquad e^{\nu} \approx \tau^{\frac{1}{n+2}} \qquad \text{and} \qquad e^{\phi} \approx \tau^{-\frac{n}{n+2}}. \tag{7.20}$$

The curvature in this region diverges and the string coupling becomes infinite. At this stage it is important to emphasize that there is a unique point along the curve no. 3 with $w'(z) = 0$ (point D on figure 8). It immediately follows from (7.2), that $\lambda(\tau)$ (and also $e^{\lambda(\tau)}$) has a minimum at this point ($\dot{\lambda} = 0$). This new feature will be discussed in section 9 in the context of confinement in the dual gauge theory. The typical form of the functions e^{λ} and e^{ν} is plotted on figure 10.

Our original goal in this section was to construct the RR perturbation of the cigar geometry. As it evident, however, from all the solutions the asymptotic form of the metric reproduces the cylinder, rather then the cigar background. In particular, we have not obtained a e^{ν} factor that monotonically decreases to zero, like in the cigar solution. It seems that in order to reach the goal, a more general ansatz for the solution of the superpotential equation is needed. For example, one may consider a superpotential of the form $W = 4e^{-\phi} w(z, \nu)$, which seems to be the most simple incorporation of the cigar solution (w depends only on ν) and the solution investigated in this section (w depends only on z).

Before closing this section let us comment on the T-dual of the background. According to the discussion in the previous section under T-duality along the circle defined by the

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coordinate θ the type-IIB metric (7.4) transforms into type-IIA metric:

$$l_s^{-2} ds^2 = e^{2\lambda(\tilde{z})} dx_{\|}^2 + \frac{d\tilde{z}^2}{(n\tilde{w}'(\tilde{z}) + \tilde{w}(\tilde{z}))^2}, \tag{7.21}$$

where $\tilde{z} = (n+1)\lambda + \varphi = 2\phi$ and $\tilde{w}$ satisfies an equation similar to (7.1):

$$(n-1)\tilde{w}'(\tilde{z})^2 + 2\tilde{w}'(\tilde{z})\tilde{w}(\tilde{z}) - \tilde{w}(\tilde{z})^2 = Q^2 e^{\tilde{z}} - c. \tag{7.22}$$

It is important to note that the type of the superpotential equation solution we have considered in the beginning of the section requires automatically that $\lambda = -\nu$. So performing the T-duality transformation we arrive at the metric compatible with our initial ansatz (2.1).

Again the equation for $\tilde{w}(\tilde{z})$ has no explicit solution for $n > 1$. For $n = 1$ we return to the BGV black hole solution discussed earlier. For $n > 1$ the numerical solutions appear on figure 8, with the coordinate z replaced by $\tilde{z} = 2\phi$. Finally, the typical form of the function $e^{2\lambda(\tilde{z})}$ repeats the result for $e^{2\lambda(z)}$ on figure 9 and figure 10. In the former case in the near horizon region ($e^{\lambda(\tilde{z})} \to 0$) the solution reduces to the AdS_n background (6.14).

8. Solutions with a non-trivial NS-NS charge

Next we would like to consider non-critical backgrounds that include H the NS-NS three form but no RR forms. In (2.7) we have assumed that H lives in the compact transverse space. That obviously implies that the dimension of the compact transverse space has to be $k = 3$. The potential associated with this case is

$$V = Q^2 e^{-2k\nu - 2\varphi} - 6e^{-2\nu - 2\varphi} - ce^{-2\varphi}. \tag{8.1}$$

Note that unlike the RR case here the potential is independent of λ. We therefore look for a superpotential in the form $W = 4e^{-\varphi} w(\nu)$. We arrive at the following differential equation for $w(\nu)$:

$$\frac{1}{3} w'^2(\nu) - w^2(\nu) = Q^2 e^{-6\nu} - 6e^{-2\nu} - c \tag{8.2}$$

and the BPS equation for ν and φ are

$$\frac{\partial \nu}{\partial \tau} = \frac{1}{3} f'(\nu), \qquad \frac{\partial \varphi}{\partial \tau} = w(\nu). \tag{8.3}$$

Using ν as the radial coordinate instead of τ we may rewrite the metric as:

$$ds^2 = dx_{\|}^2 + \frac{9}{w'^2(\nu)} d\nu^2 + e^{2\nu} d\Omega_3^2. \tag{8.4}$$

It is clear therefore that the at $\nu = \nu_0$ satisfying $w'(\nu) = 0$ there is a horizon manifold.

As another warmup exercise let us analyze first the critical case ($c = 0$). In this case the equation (8.2) has two distinct solutions:

$$w(\nu) = \frac{1}{\sqrt{2}} Q e^{-3\nu} + 3e^{-\nu} \qquad \text{and} \qquad f(\nu) = \frac{1}{\sqrt{2}} Q e^{-3\nu} - 3e^{-\nu}. \tag{8.5}$$

In the first case $w'(\nu)$ do not vanish anywhere and the corresponding metric describes a space with a boundary at $\nu \to -\infty$. In the later case $w'(\nu) = 0$ exactly at one point $\nu = \nu_0$ and substituting this function in the BPS equations (8.3) for the case of $k = 3$ we will obtain the well known NS5 solution. In particular, near $\nu = \nu_0$ it is described by the linear dilaton configuration. Unfortunately we cannot find an analytic solution of (8.2) in the non-critical case. We will fix the boundary condition at $\nu \to \infty$ by $w'(\nu) \to 0$. Then the asymptotic behavior of $w(\nu)$ at $\nu \to \infty$:

$$w(\nu) \approx -\sqrt{c} - \frac{3}{\sqrt{c}} e^{-2\nu} + \cdots \tag{8.6}$$

so that at large τ we get:

$$e^{2\nu} \approx 4c^{-\frac{1}{2}}\tau \qquad \text{and} \qquad 2\phi \approx \varphi \approx -c^{-\frac{1}{2}}\tau\,. \tag{8.7}$$

The solution (8.6) can be interpolated numerically to the finite ν region. At some point $\nu = \nu_0$ this extension will approach the curve

$$f(\nu) = -\sqrt{Q^2 e^{-6\nu} - 6e^{-2\nu} - c}\,, \tag{8.8}$$

where $f'(\nu) = 0$. We can find ν_0 by demanding that $f'(\nu)$ has a single zero at this point and therefore the space-time geodesics are complete. It results in:

$$e^{4\nu_0} = \frac{Q^2}{2} \qquad \text{and} \qquad f(\nu_0) = -\sqrt{c + 4\frac{2^{\frac{1}{2}}}{Q}} \equiv -\gamma\,. \tag{8.9}$$

Finally, at $\tau \to -\infty$ we have:

$$\nu = \nu_0 \qquad \text{and} \qquad e^{2\phi} \sim e^{-\gamma\tau}\,, \tag{8.10}$$

which is the standard linear dilaton solution one obtains in the decoupling limit of the NS5 background. The curvature is given in terms of $f(\nu)$ by:

$$\alpha'\mathcal{R} = 2w''w' - \frac{4}{3}w'^2 - 6e^{-2\nu_0}\,. \tag{8.11}$$

As expected it goes to zero at large ν ($\tau \to \infty$) and at for $\nu \to \nu_0$ ($\tau \to -\infty$) one finds $\alpha'\mathcal{R} \approx 6\sqrt{2}/Q$.

Again one can derive the near horizon solution directly from the second order differential equations:

$$\partial_\rho^2 \lambda = 0\,, \qquad \partial_\rho^2 \nu - 2e^{-2\nu - 2\varphi} + Q^2 e^{-6\nu - 2\varphi} = 0 \tag{8.12}$$

and

$$\partial_\rho^2 \varphi + ce^{-2\varphi} + 6e^{-2\nu - 2\varphi} - Q^2 e^{-6\nu - 2\varphi} = 0\,. \tag{8.13}$$

9. Holographic dual gauge theories

The gauge/gravity holographic duality relates a gravitational background in ten dimensions to a gauge theory in the large-N limit residing on the boundary of the background space-time [32, 33, 34]. It is believed that the full boundary gauge theory, namely, not only its planar limit is dual to the full string theory. For backgrounds with a constant dilaton, like in the $AdS_5 \times S^5$ case, the corresponding boundary field theory is a CFT whereas backgrounds with a non-constant dilaton map into non-conformal boundary field theories. It was conjectured in [7] that this concept holds also for non-critical SUGRA backgrounds and even for non-supersymmetric backgrounds that include gravity. In this section we explore the holographic duality of the non-critical solutions described in the previous sections.

Before dwelling into the conjectured dual theories let us first develop some intuition from brane configurations that associate with the SUGRA backgrounds we discuss. In critical dimensions a useful way to understand the D_p brane SUGRA backgrounds [31] is as follows. Consider first a space-time with flat metric, constant dilaton and no additional forms. Now place a stack of N D_p branes. The back-reaction of the latter transforms the background into that of D_p branes, namely, a curved space-time with a non-constant dilaton (apart from the $p = 3$ case) and a $p + 2$ RR form with a flux which is equal to N. Upon taking the near horizon limit one ends up in backgrounds like for instance the $AdS_5 \times S^5$ for $p = 3$. In non-critical backgrounds one obviously cannot start with a flat $d \neq d_{\text{critical}}$ dimensional space time, and a constant dilaton and no forms. Instead one starts with a flat d dimensional Minkowski space-time with a linear dilaton which was shown in (4.8) to be a solution of the equations of motions. The back-reaction of adding N D_p branes generates the $AdS_{p+2} \times S^{d-p-2}$ backgrounds with $p + 2$ RR forms which again have N units of flux. However, unlike the critical cases, here the dilaton is constant for any p.

A class of backgrounds with non-constant dilaton were described in section 7 associated with manifolds that include an S^1 factor. These backgrounds can also be thought of the back-reaction of N D_p branes placed in manifolds of $R^{1,p+1} \times S^1$ geometry with a linear dilaton which is also a solution (4.8). Recall that this background is equivalent according to [3, 4, 5] to the $\mathcal{N} = 2$ super Liuville theory which is the world sheet description of non-critical strings.

We now proceed to describe the basic properties of the holographic dual field theories. We start with features that are shared by all the different classes of background solutions and then we describe each class separately.

9.1 The entropy and the duality to gauge degrees of freedom

The basic concept of holography is that the physics of a bulk space that includes gravity can be described by a field theory that lives on the boundary of this space. If the boundary field theory is a U(N) or SU(N) gauge theory, the number its degrees of freedom at the UV regime scales like N^2. To be more precise one has to introduce a UV cutoff δ and the entropy of the gauge theory scales like

$$S_{\text{gauge}} \sim \frac{N^2}{\delta^3}\,, \tag{9.1}$$

where the factor of $1/\delta^3$ is the number of cells one has to divide the volume of the three sphere on which the field theory is defined.

Thus to verify the holographic duality one has to show that the entropy of the SUGRA backgrounds, that are claimed to be duals of gauge theories, scale in the same manner with N and δ. To estimate the entropy of the SUGRA bulk theory we use the Bekenstein-Hawking bound which is the area in Planck units, namely, $S_{\text{SUGRA}} = \frac{\text{Area}}{4G_N}$. One way to evaluate this bound is to determine the dependence of the area and G_N on N in the string frame. Since in this frame the metric is N independent, so is the area. On the other hand

$$G_N \sim e^{2\phi} \sim N^{-2} \quad \to \quad S_{\text{SUGRA}} \sim N^2\,. \tag{9.2}$$

Thus indeed the entropy scales as N^2. Another way is to compute the area in units of G_N. The area of the boundary diverges and similarly to the field theory calculation a cutoff has to be introduced. For instance if one uses the following parameterization of an AdS space

$$ds^2 = \frac{dx_{\|}^2 + dz^2}{z^2}\,, \tag{9.3}$$

then the boundary is at $z = 0$. Introducing a cutoff implies that the boundary is taken at $z \sim \delta$. To compute the area in Planck units we have to switch from the string frame which we have used so far to the Einstein frame. The two frames are related through

$$g_{ij}^{(E)} = e^{-\frac{4}{d-2}\phi}\, g_{ij}^{(s)}\,. \tag{9.4}$$

Thus since $e^{\phi} \sim \frac{1}{N}$ (5.2) it follows that in the Einstein frame the radii of the AdS_{n+1} and the S^k part are proportional to $N^{\frac{2}{d-2}}$, so that the area relevant for the calculation of the entropy bound goes like:

$$\begin{aligned} S_{\text{SUGRA}} &\sim \text{Area} \sim V_{S^k}\left(\frac{R_{AdS}}{\delta}\right)^{n-1} \sim (R_{S^k})^k \left(\frac{R_{AdS}}{\delta}\right)^{n-1} \\ &\sim N^{\frac{2}{d-2}(k+n-1)}\delta^{-(n-1)} \sim N^2\delta^{-(n-1)}\,, \end{aligned} \tag{9.5}$$

where δ is the UV cutoff and we have used the fact that $d = n + 1 + k$. In particular in four dimensions with $n = 4$ we find an agreement with S_{gauge}. Hence we have shown that indeed the bound on the entropy of the SUGRA theory scales in the same way as the boundary gauge theory.

So far we have addressed the conformal backgrounds described in section (5). What about the RR deformed cylinder backgrounds of section 7? It turns out that the same result applies also to that case. Recall, (5.2) that for this case too the dilaton scales like $e^{\phi} \sim 1/N$ and hence again the radii are proportional to $N^{\frac{2}{d-2}}$ and thus (9.5) holds also for the RR deformed cylinder backgrounds.

9.2 A novel large-N limit

In the original AdS/CFT duality in ten dimensions, the duality maps the bulk physics into that of a boundary gauge theory with large N and $g_s N \sim g_{\rm YM}^2 N \gg 1$. The latter condition follows from the requirement to have a small scalar curvature of the background since otherwise higher order α' corrections are not negligible. This large-N limit is clearly different form the perturbative one where N is taken to infinity such that $g_{\rm YM}^2 N$ is finite and small: $g_{\rm YM}^2 N < 1$.

Let us now find out what is the consistent region of the gauge theories associated with the conformal backgrounds discussed in section 5 and the RR deformed cylindrical backgrounds of section 7. Again the SUGRA approximation holds only provided that the scalar curvature is small. According to (5.7) the radii of the AdS_{n+1} and of the S^k are Q (and hence N) independent constants of order unity and so that the curvature is

$$\alpha' \mathcal{R} = -c\,. \tag{9.6}$$

Hence unlike the critical AdS/CFT duality, here the curvature is fixed, of order unity and cannot be reduced by taking a large-N limit.

There is yet another difference between the non-critical conformal solutions and the $AdS_5 \times S^5$ case. In both cases the SUGRA approximation is valid only provided the string coupling is small. However, whereas in the latter case the dilaton is constant independent of N, in the non-critical case (5.2) the string coupling

$$g_s \sim e^{\phi_0} = \left[\frac{1}{n+1-k}\left(\frac{(n+1-k)(k-1)}{c}\right)^k \frac{2c}{Q^2}\right]^{1/2} \tag{9.7}$$

and therefore small string coupling means large N. Moreover, if we adopt the conventional correspondence between g_s and $g_{\rm YM}^2$ then since $g_s \sim \frac{1}{N}$, we find that the 't Hooft coupling (5.9) is a constant of order unity

$$g_{\rm YM}^2 N \sim \left(\frac{2c}{n+1-k}\left(\frac{(n+1-k)(k-1)}{c}\right)^k\right)^{1/2}. \tag{9.8}$$

To summarize, the large-N limit that one has to take in the boundary gauge theory dual to the non-critical SUGRA is different than the one taken in the critical case:

$$\begin{aligned} \text{critical}: \quad & N \to \infty\,, \quad g_{\rm YM}^2 N \gg 1 \\ \text{non-critical}: \quad & N \to \infty\,, \quad g_{\rm YM}^2 N \sim 1\,. \end{aligned} \tag{9.9}$$

It is interesting to note that the large-N limit that one has to take in the gauge theories which are the duals of the non-critical SUGRAs is in between the perturbative large-N limit with $g_{\rm YM}^2 N < 1$ and the limit in the critical cases which is $g_{\rm YM}^2 N \gg 1$.

9.3 The gauge theories duals of the $AdS_{p+2} \times S^{d-p-2}$ SUGRA backgrounds

The dual field theories associated with $AdS_{p+2} \times S^{d-p-2}$ should be, following the discussion in subsection 9.1, conformal or superconformal gauge theories. The conformal invariance is obviously due to the constant dilaton that maps into a scale invariant gauge coupling.

As for supersymmetry, the background solutions do solve the BPS equations. However, as was mentioned above this last fact is not a proof of supersymmetry but it is a necessary condition. It is also important to note that the dual gauge theories are at a fixed points in the strong 't Hooft coupling regime since as discussed above the SUGRA corresponds to the gauge theory with $g^2_{\mathrm{YM}}N \sim 1$.

According to [1] a necessary condition for eliminating the tachyons of the non-critical strings is if they admit space-time supersymmetry which can occur only if they reside in even space-time dimensions. Adopting this condition to our non-critical models, means that the dual gauge theories can live on boundaries of bulk space-time of 2,4,6 and 8 dimensions. The gauge theories have global symmetries associated with the isometries of the various backgrounds. In the following table we list in for each world volume dimensions the possible SUGRA models and their corresponding global symmetries.

Gauge theory in n dimensions	The SUGRA manifold	The global symmetry
2	$AdS_3 \times S^5$	SO(6)
3	AdS_4	–
4	$AdS_4 \times S^2$	SO(3)
4	$AdS_5 \times S^3$	SO(4)
5	AdS_6	–
5	$AdS_6 \times S^2$	SO(3)
7	AdS_8	–

Table 1: The various SUGRA backgrounds and their dual gauge theories.

If due to a yet unknown mechanism the theories with odd d can also be stabilized then there there are additional possible dual gauge fields like the AdS_5 and the associated four dimensional gauge theory with no global symmetry.

In order to identify the gauge theories given in the table with known superconformal gauge theories one has to do certain additional checks that we leave for future investigation. In particular one should determine the amount of supersymmetry in the the various gravitational backgrounds. From a brief glance on the table it seems that the two models with SO(3) isometry may correspond to superconformal gauge theories. It is known [35] that in three dimensions a theory with $\mathcal{N}$ supersymmetries has an R symmetry of SO(N). Hence the $AdS_4 \times S^2$ model may correspond to $\mathcal{N} = 3$ in three dimensions. There is a five dimensional superconformal theory with $SP(1)$ R symmetry. This may relate to the $AdS_6 \times S^2$ model.

For the rest of the models we cannot relate the data given in the table with known superconformal gauge theories. There are several logical explanations to this situation: (i) It might be that incorporating higher curvature correction will change for instance the isometry of the transverse manifold and hence the global symmetry of the gauge theories. (ii) It might be that the dual gauge theories are non-supersymmetric theories. For instance, one could imagine four dimensional theories with four additional matter fields in the adjoint that admit the SO(4) global symmetry and strongly coupled fixed points. (iii) It might be that only part of the full isometry translates into an R symmetry or a global symmetry of the gauge theory due to the fact, that the GSO projection is compatible only with a subgroup of the full isometry group. Such a case occurs for the $AdS_3 \times S^3$ [36]. One may imagine a situation where only the subgroup $\mathrm{SU}(2) \times \mathrm{U}(1)$ of the SO(4) isometry group survives the GSO projection and hence a potential compatibility with the R symmetry of $\mathcal{N} = 2$.

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Assuming that there are conformal gauge theories that correspond to these non-critical SUGRA backgrounds, one can turn on the known machinery of computing the conformal dimensions of chiral operators computing correlation functions etc. in a similar manner to what was done in the critical cases.

9.4 The AdS black hole solutions and their gauge theory duals

Let us now examine the gauge theory duals of the AdS black hole solutions discussed in subsection 5.2. Note that the latter have a thermal factor in the metric that resembles that of near extremal critical D_p branes [31]. However whereas in the former case the thermal factor is $1-(\frac{u_0}{u})^n = 1-(\frac{u_0}{u})^{p+1}$ in the latter case it is $1-(\frac{u_0}{u})^{7-p}$. Remarkably in the four dimensional case ($n=4, p=3$) the non-critical AdS_5 black hole solution matches that of the black hole of $AdS_5 \times S^5$. (We refer to the AdS part, obviously they have different S^k). However there is still one difference between the two cases and that is the AdS radius:

$$\text{critical}: \quad R_{AdS} = \left(g_{\text{YM}}\sqrt{N}\right)^{1/2} \tag{9.10}$$

$$\text{non-critical}: \quad R_{AdS} = \left(\frac{n(n+1-k)}{c}\right)^{1/2}. \tag{9.11}$$

Due to the similarity between the critical and non-critical near extremal solutions, we do not have to redo the calculations in the gauge theory but rather read them from the known results of the near extremal $AdS_5 \times S^5$ solution. In particular, we can implement the idea proposed in [37] of imposing anti-periodic boundary conditions while taking the large temperature limit, namely small thermal radius of the background, which leads to a pure YM theory in three dimensions. The properties of the 3d gauge dynamics that we will focus on are the Wilson loop and the glue-ball spectrum.

To determine the Wilson loop one can write down the NG action associated with the background metric and determine the classical configuration of the string [38, 39]. Instead we can use the general result of [40, 41, 42]. According to these results if one of the following two conditions is obeyed, the corresponding Wilson line admits an area law behavior:

$$\begin{aligned}
&g_{00}g_{ii}(\tau) \quad \text{has a minimum at} \quad \tau_{\min} \quad \text{with} \quad g_{00}g_{ii}(\tau_{\min}) > 0\,, \\
&g_{00}g_{\tau\tau}(\tau) \quad \text{diverges at} \quad \tau_{\text{div}} \quad \text{with} \quad g_{00}g_{ii}(\tau_{\text{div}}) > 0\,.
\end{aligned} \tag{9.12}$$

It is easy to check that after the reduction to 3d $g_{00}g_{uu} = [1-(\frac{u_0}{u})^4]^{-1}$ and it diverges at $u=u_0$. The conclusion is therefore that indeed the Wilson loop in this background admits an area law behavior with string tension of

$$\text{string tension} = \frac{1}{2\pi}\left(\frac{u_0}{R_{AdS}}\right)^2 = \frac{1}{2}\pi\frac{8}{c}T^2\,, \tag{9.13}$$

where T is the temperature which is related to u_0 as follows $u_0 = \pi R_{AdS}^2 T$. It is interesting to compare this result to the one derived from the critical AsS black hole. In the latter case the factor $8/c$ is replaced by $\sqrt{g_{\text{YM}}^2 N}$ so that one has later to interpolate between this result and the one associated with finite $g_{\text{YM}}^2 N$. In our case this last step is obviously unnecessary.

To describe the four dimensional Wilson loop, one can start [37] with the conformal background of M^5 branes, namely $AdS_7 \times S^4$, introduce two circles and in the limit of small radii the world volume manifold is reduced from a six dimensional one to four dimensions. Alternatively, one can take the infinite temperature limit of the non-conformal D_4 SUGRA background [38, 39]. Here using the non-critical solutions we can start with the near extremal AdS_6 solution, introduce temperature, and take the infinite temperature limit thus getting a four dimensional theory. The outcome of the calculation of the string tension will be the same as above (changing the factor 8 into 30). Note that there is a difference between that and the critical case where the 4d string tension is $\frac{1}{2\pi}(\frac{u_0}{R_{AdS}})^{3/2}$.

Next we consider the glue-ball spectrum. The analysis of the three dimensional glue-balls is identical to the one done in the near extremal $AdS_5 \times S^5$ background. Solving the equation of motion of the dilaton in the non-critical near extremal background, choosing the normalizable solution, imposing regularity at the horizon, one finds a discrete spectrum with a mass gap. For instance the masses that correspond to the 0^{++} glue-balls are $M^2_{0^{++}} \sim 1/T^2$. Since this result is independent of R_{AdS} it is identical to the one in the critical case. However there is a physical difference between the picture in the non-critical and critical cases since in the latter case the string tension is proportional to $\sqrt{g^2_{\rm YM}N}$, $M^2_{0^{++}}$ does not scale like the string tension, whereas in our case it does.

To get the glue-ball spectrum in four dimensions one has to solve the equation of motion of the dilaton in the background achieved from that of the near extremal AdS_6 in the zero radius limit. Assuming the fluctuation of the dilaton is $\delta\phi = \tilde{\phi}(u)e^{ikx}$, the laplacian that enters the equation, which determines the glue-ball spectrum

$$\partial_u[u(u^5 - u_0^5)\partial_u]\tilde{\phi}(u) + M^2 u\tilde{\phi}(u)\,, \qquad \text{where} \qquad M^2 = -k^2 \tag{9.14}$$

is different than the one solved in the critical case.

9.5 The gauge duals of the RR deformed cylinder SUGRA backgrounds

The RR deformed cylindrical solutions were found for any world-volume dimension n. Let us concentrate on the four dimensional case. As discussed above from the entropy of the SUGRA it seems natural that the boundary gauge theory has an SU(N) gauge symmetry. Since the dilaton is varying the dual gauge theory is non-conformal. As was stated above we have not analyzed the amount of supersymmetry these backgrounds admit. However if we relate to the naive brane configuration constructed on the $R^{1,3} \times cigar$ we would conclude that the dual four dimensional gauge theory is $\mathcal{N} = 1$ supersymmetric. By construction these solutions have an SO(2) isometry associated with the S^1 part of the manifold and hence one expects that the dual gauge theory has an U$_R$(1) symmetry. Combining these three ingredients of the local symmetry group, the supersymmetry and the R symmetry, indicates that the dual gauge theory maybe a close cousin of the $\mathcal{N} = 1$ SYM theory. One clear difference this theory and the dual of our SUGRA backgrounds is the U$_R$(1) symmetry since in our SUGRA it is a unbroken symmetry whereas in $\mathcal{N} = 1$ SYM theory it is broken both by instantons and also spontaneously. To look for backgrounds that correspond to a spontaneously broken symmetry one has to abandon our original ansatz where all the fields depend only on the radial direction and not on the θ (the coordinate along the S^1) direction. Modifying the ansatz will render the task of finding solutions to BPS equations much more complicated so we left it for future investigation.

If indeed the dual gauge theory is a relative of the $\mathcal{N}=1$ SYM theory, it should admit confinement. As was discussed above this can be checked both by the determination of the expectation value of the Wilson loop as well as by a computation of the glue-ball spectrum. Using the necessary conditions to have confining Wilson loop given above it is easy to see that the first condition translates into having a minimum to e^{λ} at a non vanishing value. A convenient way to check if this condition is obeyed is to examine figure 10.

It is clear that apart from the class of solutions described by the line that ends on point A the rest of the solutions do admit a minimum for

$$\partial_\tau e^{\lambda}(\tau_{\min}) = 0 \qquad e^{\lambda}(\tau_{\min}) > 0\,. \tag{9.15}$$

Thus according to the above criterion the gauge theory dual to the RR deformed SUGRA backgrounds indeed admits confinement behavior. Note however that unlike in critical duals of confining gauge theories that are close relatives of $\mathcal{N}=1$ SYM theory like those of [43, 44], in the present case the minimum of e^{λ} does not occur at the "end" of the radial direction but rather at some mid point which means that in fact the bulk space-time has two boundaries. Backgrounds of this form were found also in [16]. Since beyond $\tau_{\min}$ the validity of the SUGRA approximation is doubtful we will not attempt to discuss the background at that region. Generically if indeed there are two boundaries one has to be worried about loss of unitarity on the boundary field theory due to "leakage" of signals to the other boundary. As was mentioned above a background which is a faithful dual of the $\mathcal{N}=1$ SYM theory, has to admit a spontaneous $\mathrm{U}_R(1)$ breaking as well as a breaking due to instantons. These phenomena cannot take place using our ansatz of dependence only on the radial coordinate. One may anticipate that solutions that share a breaking of the $\mathrm{U}_R(1)$ will have a different radial dependence so that the problem of two boundaries will be avoided.

10. Summary and discussion

There are several reasons to believe that the string theory of QCD is not a ten dimensional theory. KK states that cannot be decoupled from the physical hadronic spectrum is one such a phenomenon. This situation calls for string theory in less than ten dimension. Due to our ignorance about quantizing string theories on RR backgrounds, in the NSR formulation we cannot do much in the study of non-critical string theories that may be candidates for describing hadronic physics. This pushes us to address first the small α' limit of string theories, namely the low energy SUGRA solutions. That was precisely the aim of this paper.

The equations of motion that follow from the requirement for vanishing of the β function on the world sheet of the string, are in general a set of complicated coupled second order partial differential equations. The equations for non-critical strings are even more complicated due to an additional term in the potential. To simplify the system we choose a particular ansatz of the metric which is geared toward holographic applications. This ansatz assumes dependence only on a radial direction thus turning the problem into that of ordinary differential equations. A further technical simplification is achieved by converting the problem into BPS first order differential equations using the superpotential formalism. From all possible topologies we have focused on solution of two types: backgrounds with a structure of $AdS_{p+2} \times S^{d-p-2}$ and those that asymptote a linear dilaton with a topology of $R^{1,d-3} \times R \times S^1$. Whereas the former class are analytic solutions the latter include numerical solutions with approximated analytic behavior close to the origin.

The SUGRA solutions we have found can serve as an anti holographic description of gauge theories in a particular large-N limit which is neither the perturbative nor that of the critical AdS correspondence. It is characterized by $N \to \infty$ and $g^2_{\rm YM} N \sim 1$. We have made the first steps of the analysis of the properties which are of interest from the gauge dynamics point of view like the Wilson loops and the glue-ball spectra.

Let us now propose several open questions that deserve further investigation:

- A very essential issue that we have not addressed in this paper is controlling the higher curvature corrections. Generically we do not have yet any evidence that the latter are small and hence it will be interesting to compute the impact of the leading order curvature correction on the backgrounds discussed. Our belief is that the general structure of the background space-time will not be changed but the characterizing parameters will be modified, that is to say that for instance the $AdS_{n+1} \times S^k$ structure will remain but the radii of the AdS_{n+1} and S^k will be modified.

- In the search of solutions of the equation of motions of the non-critical low energy effective action (2.2) we did not take the most general ansatz both for the metric and for the various forms. One can extend our search by looking for (i) more general metric in particular other possible transverse spaces like the conifold, (ii) solutions with NS form combined with a RR form, (iii) solutions with more than one RR form, (iv) solutions where the metric, dilaton and the form depend on other coordinates rather only the radial one.

- The full analysis of the properties of the holographic dual gauge theories has to be performed. In particular determining the glue-ball spectrum quantum corrections of the Wilson loop, Baryons etc. (see [45, 46]) in the context of the confining backgrounds like the AdS black hole solutions and the RR deformed cylindrical solutions.

- One established route to go beyond the SUGRA limit is to invoke the Penrose limit and quantize the plane wave string theory. Hopefully it can be done for the class of the $AdS_{n+1} \times S^k$ backgrounds and the corresponding black hole solutions. An interesting challenge is to apply the limit also for the solutions that were found only numerically.

- To incorporate dynamical quark in the dual gauge theories, one may use the idea of introducing probes similar to the probes introduced for confining critical backgrounds [47] and deducing the mesonic spectrum from the spectrum of the probes fluctuations.

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A. The derivation of the quantum mechanical action

For the metric

$$l_s^{-2} ds^2 = d\tau^2 + e^{2\lambda} dx_{||}^2 + e^{2\nu} d\Omega_k^2 \,,$$

we calculate

$$\begin{aligned} \alpha' \mathcal{R} &= -2n\lambda'' - n(n+1)(\lambda')^2 + k(k-1)e^{-2\nu} - \\ &\quad -2k\nu'' - 2kn\nu'\lambda' - k(k+1)(\nu')^2 \,, \\ \sqrt{G} &= e^{n\lambda + k\nu} \,. \end{aligned} \tag{A.1}$$

Thus

$$\begin{aligned} S = \int \sqrt{G}\,\mathcal{R} &= {\alpha'}^{-1} \int e^{n\lambda+k\nu} \big[(2n\lambda' + 2k\nu')(n\lambda' + k\nu') - n(n+1)(\lambda')^2 + \\ &\qquad + k(k-1)e^{-2\nu} - 2kn\nu'\lambda' - k(k+1)(\nu')^2\big] \\ &= {\alpha'}^{-1} \int e^{n\lambda+k\nu} \left[n(n-1)(\lambda')^2 + k(k-1)e^{-2\nu} + 2kn\nu'\lambda' + k(k-1)(\nu')^2\right] \\ &= {\alpha'}^{-1} \int e^{n\lambda+k\nu} \left[(n\lambda' + k\nu')^2 - n(\lambda')^2 + k(k-1)e^{-2\nu} - k(\nu')^2\right] . \end{aligned} \tag{A.2}$$

B. From the BPS equations to the equations of motion

The BPS equations take the form

$$f_a{}' = \frac{1}{2} G_{ab} \partial^b W \,. \tag{B.1}$$

Differentiating with respect to τ we get

$$\begin{aligned} f_a{}'' &= \frac{1}{2}[\partial_c G_{ab} f^{c\prime} \partial^b W + G_{ab} \partial_c \partial^b W f^{c\prime}] \\ &= \partial_c G_{ab} f^{c\prime} f^{b\prime} + \frac{1}{4} G^{cd} \partial_a \partial_c W \partial_d W \\ &= \partial_c G_{ab} f^{c\prime} f^{b\prime} + \frac{1}{8} \partial_a (G^{cd} \partial_c W \partial_d W) - \frac{1}{8} \partial_a (G^{cd} \partial_c W \partial_d W) \\ &= \partial_c G_{ab} f^{c\prime} f^{b\prime} + \partial_a V - \frac{1}{2} \partial_a G^{bc} f_c{}' f_b{}' \\ &= \partial_c G_{ab} f^{c\prime} f^{b\prime} + \partial_a V + \frac{1}{2} \partial_a G_{bc} f^{c\prime} f^{b\prime} \end{aligned} \tag{B.2}$$

and the last line is indeed the equation of motion (2.12).

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C. Action and BPS equations in Einstein metric

Define the Einstein metric

$$ds_E^2 = e^{-\frac{4}{n+k-1}\phi} ds^2 \tag{C.1}$$

where ds^2 is the string frame metric, and parameterize the Einstein metric for $k \neq 0$ as follow

$$ds_E^2 = e^{\frac{-2k}{n-1}B}\left(du^2 + e^{\frac{8}{n}A(u)}dx_{||}^2\right) + e^{2B(u)}d^2\Omega_k \tag{C.2}$$

so that the following relations hold

$$\frac{4}{n+k-1}\phi - \frac{2k}{n-1}B + \frac{8}{n}A = 2\lambda; \qquad \frac{4}{n+k-1}\phi + 2B = 2\nu\,; \tag{C.3}$$

and

$$d\tau = du\, e^{\frac{4}{n+k-1}\phi - \frac{2k}{n-1}B}\,, \tag{C.4}$$

then the corresponding action has the form of (2.19)

$$\begin{aligned} S &= \int du e^{4A}\left(3(A')^2 - \frac{3}{16}\frac{kn(n+k-1)}{(n-1)^2}(B')^2 - \frac{3}{4}\frac{n}{(n-1)(n+k-1)}(\phi')^2 - V_E(f)\right) \\ V_E &= \frac{3}{16}\frac{n}{n-1}\left[-e^{-\frac{2k}{n-1}B}(ce^{\frac{4}{n+k-1}\phi} + \Lambda e^{\frac{2(n+k+1)}{(n+k-1)}\phi}) + k(k-1)e^{-\frac{(n+k-1)}{(n-1)}B} + V_{\rm RR}\right] \\ V_{\rm RR} &= \frac{1}{4}N^2 e^{2\phi - \frac{2(n+k-1)}{(n-1)}B}\,, \end{aligned} \tag{C.5}$$

which implies that $G^{\phi\phi} = \frac{2}{3}\frac{(n-1)(n+k-1)}{n}$ and $G^{BB} = \frac{8}{3}\frac{(n-1)^2}{kn(n+k-1)}$.

D. The cigar solution in the Einstein frame

Another superpotential that corresponds to the potential

$$V_E = -\frac{3}{16}c\frac{n}{(n-1)}e^{(\frac{4}{n}\phi - \frac{2}{n-1}B)} \tag{D.1}$$

takes the form

$$W_E = -\frac{3}{8}\frac{n}{n-1}\sqrt{c}\left[e^{\frac{4}{n}\phi + \frac{n-2}{n-1}B} + e^{-\frac{n}{n-1}B}\right] \tag{D.2}$$

Defining now a new radial coordinate $dr = 2\sqrt{c}\, e^{\frac{2}{n}\phi - \frac{1}{n-1}B}du$ the BPS equations read

$$\begin{aligned} \partial_r B &= -\left[\frac{(n-2)}{n}e^{\frac{2}{n}\phi + B} - e^{-\frac{2}{n}\phi - B}\right] \\ \partial_r \phi &= -\left[e^{\frac{2}{n}\phi + B}\right] \\ \partial_r A &= \frac{1}{8}\frac{n}{n-1}\left[e^{\frac{2}{n}\phi + B} + e^{-\frac{2}{n}\phi - B}\right] \end{aligned} \tag{D.3}$$

These equations have the following solution

$$\begin{aligned} \phi = -\ln(\cosh(r))\,, \qquad B &= \ln(\sinh(r)) - \frac{n-2}{n}\ln(\cosh(r)) \\ A &= \frac{n}{4(n-1)}\ln(\cosh(r)\sinh(r))\,, \end{aligned} \tag{D.4}$$

which means that

$$e^{\frac{4}{n}\phi+2B} = \tanh^2 r \qquad e^{\frac{4}{n}\phi-\frac{2}{n-1}B+\frac{8}{n}A} = 1\,, \tag{D.5}$$

so that the metric is that of a cigar, namely:

$$ds^2 = dr^2 + dx_{||}^2 + \tanh^2(r)d^2\theta\,. \tag{D.6}$$

E. The BGV solution from the equations of motion

Using the facts that $d\tau = \frac{d\phi}{\sqrt{l(\phi)}}$ and $d\tau = e^{-\varphi}d\rho$ we can re-write the equations of motion (2.9) for $n = 1$ and $k = 0$ in the formulation, where the dilaton is the radial direction:

$$\partial_\phi^2\lambda + 2\partial_\phi\lambda(\partial_\phi\lambda - 1) - \frac{Q^2}{2}e^{-2(\lambda-\phi)} = 0 \tag{E.1}$$

and

$$(\partial_\phi\lambda - 1) + \frac{c}{4}e^{-2\lambda} - \frac{Q^2}{2}e^{-2(\lambda-\phi)} = 0\,. \tag{E.2}$$

The solution to the first equation can be obtained as follows. First we write:

$$\begin{aligned}\partial_\phi^2\left(e^{2\lambda-\phi}\right) &= 2e^{2\lambda-\phi}(\partial_\phi^2\lambda + \frac{1}{2}(2\partial_\phi\lambda - 1)^2)\\ &= 2e^{2\lambda-\phi}\left(\partial_\phi^2\lambda + 2\partial_\phi\lambda(\partial_\phi\lambda - 1) + \frac{1}{2}\right)\\ &= -Q^2e^\phi + e^{2\lambda-\phi}.\end{aligned}$$

Setting $\eta \equiv e^{2\lambda-\phi}$ we obtain

$$\eta'' - \eta = -Q^2e^\phi$$

and the solution is

$$\eta = -\frac{Q^2}{2}(\phi + C_0)e^\phi + C_1e^{-\phi}\,.$$

Thus

$$e^{2\lambda} = -\frac{Q^2}{2}(\phi + C_0)e^{2\phi} + C_1\,.$$

Finally, substituting this solution into the dilaton equation we obtain that $C_1 = \frac{c}{4}$.

References

[1] D. Kutasov and N. Seiberg, *Noncritical superstrings*, *Phys. Lett.* **B 251** (1990) 67.

[2] D. Kutasov, *Some properties of (non)critical strings*, hep-th/9110041.

[3] A. Giveon, D. Kutasov and O. Pelc, *Holography for non-critical superstrings*, *J. High Energy Phys.* **10** (1999) 035 [hep-th/9907178].

[4] A. Giveon and D. Kutasov, *Little string theory in a double scaling limit*, *J. High Energy Phys.* **10** (1999) 034 [hep-th/9909110].

[5] A. Giveon and D. Kutasov, *Comments on double scaled little string theory*, *J. High Energy Phys.* **01** (2000) 023 [hep-th/9911039].

JHEP07(2004)049

[6] K. Hori and A. Kapustin, *Duality of the fermionic 2d black hole and $N = 2$ Liouville theory as mirror symmetry*, *J. High Energy Phys.* **08** (2001) 045 [hep-th/0104202].

[7] A.M. Polyakov, *The wall of the cave*, *Int. J. Mod. Phys.* **A 14** (1999) 645 [hep-th/9809057].

[8] I.R. Klebanov and A.A. Tseytlin, *D-branes and dual gauge theories in type 0 strings*, *Nucl. Phys.* **B 546** (1999) 155 [hep-th/9811035].

[9] G. Ferretti and D. Martelli, *On the construction of gauge theories from non critical type 0 strings*, *Adv. Theor. Math. Phys.* **3** (1999) 119 [hep-th/9811208].

[10] G. Ferretti, J. Kalkkinen and D. Martelli, *Non-critical type 0 string theories and their field theory duals*, *Nucl. Phys.* **B 555** (1999) 135 [hep-th/9904013].

[11] I.R. Klebanov and A.A. Tseytlin, *Asymptotic freedom and infrared behavior in the type 0 string approach to gauge theory*, *Nucl. Phys.* **B 547** (1999) 143 [hep-th/9812089].

[12] M.R. Garousi, *String scattering from D-branes in type 0 theories*, *Nucl. Phys.* **B 550** (1999) 225 [hep-th/9901085].

[13] J.A. Minahan, *Asymptotic freedom and confinement from type 0 string theory*, *J. High Energy Phys.* **04** (1999) 007 [hep-th/9902074].

[14] N. Nekrasov and S.L. Shatashvili, *On non-supersymmetric CFT in four dimensions*, *Phys. Rept.* **320** (1999) 127 [hep-th/9902110].

[15] M. Billò, B. Craps and F. Roose, *On D-branes in type 0 string theory*, *Phys. Lett.* **B 457** (1999) 61 [hep-th/9902196].

[16] A. Armoni, E. Fuchs and J. Sonnenschein, *Confinement in 4d Yang-Mills theories from non-critical type 0 string theory*, *J. High Energy Phys.* **06** (1999) 027 [hep-th/9903090].

[17] Y. Imamura, *Branes in type 0/type-II duality*, *Prog. Theor. Phys.* **102** (1999) 859 [hep-th/9906090].

[18] K. Ghoroku, *Yang-Mills theory from non-critical string*, *J. Phys.* **G 26** (2000) 233 [hep-th/9907143].

[19] A. Maloney, E. Silverstein and A. Strominger, *de Sitter space in noncritical string theory*, hep-th/0205316.

[20] G. Papadopoulos and A.A. Tseytlin, *Complex geometry of conifolds and 5-brane wrapped on 2-sphere*, *Class. and Quant. Grav.* **18** (2001) 1333 [hep-th/0012034].

[21] V. Borokhov and S.S. Gubser, *Non-supersymmetric deformations of the dual of a confining gauge theory*, *J. High Energy Phys.* **05** (2003) 034 [hep-th/0206098].

[22] S. Kuperstein and J. Sonnenschein, *Analytic non-supersymmtric background dual of a confining gauge theory and the corresponding plane wave theory of hadrons*, *J. High Energy Phys.* **02** (2004) 015 [hep-th/0309011].

[23] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, *Phys. Rept.* **244** (1994) 77 [hep-th/9401139].

[24] N. Berkovits, S. Gukov and B.C. Vallilo, *Superstrings in 2d backgrounds with RR flux and new extremal black holes*, *Nucl. Phys.* **B 614** (2001) 195 [hep-th/0107140].

[25] O. Aharony, M. Berkooz, D. Kutasov and N. Seiberg, *Linear dilatons, NS5-branes and holography*, *J. High Energy Phys.* **10** (1998) 004 [hep-th/9808149].

[26] C. Kounnas, *Construction of superstrings in wormhole like backgrounds*, hep-th/9402080.

[27] R.C. Myers, *New dimensions for old strings*, *Phys. Lett.* **B 199** (1987) 371.

[28] E. Alvarez, C. Gomez and L. Hernandez, *Non-critical Poincaré invariant bosonic string backgrounds and closed string tachyons*, *Nucl. Phys.* **B 600** (2001) 185 [hep-th/0011105].

[29] C.G. Callan Jr., J.A. Harvey and A. Strominger, *Supersymmetric string solitons*, hep-th/9112030.

[30] S. Murthy, *Notes on non-critical superstrings in various dimensions*, *J. High Energy Phys.* **11** (2003) 056 [hep-th/0305197].

[31] N. Itzhaki, J.M. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and the large-N limit of theories with sixteen supercharges*, *Phys. Rev.* **D 58** (1998) 046004 [hep-th/9802042].

[32] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[33] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[34] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[35] S. Minwalla, *Restrictions imposed by superconformal invariance on quantum field theories*, *Adv. Theor. Math. Phys.* **2** (1998) 781 [hep-th/9712074].

[36] A. Giveon and M. Roček, *Supersymmetric string vacua on $AdS_3 \times N$*, *J. High Energy Phys.* **04** (1999) 019 [hep-th/9904024].

[37] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131].

[38] A. Brandhuber, N. Itzhaki, J. Sonnenschein and S. Yankielowicz, *Wilson loops in the large-N limit at finite temperature*, *Phys. Lett.* **B 434** (1998) 36 [hep-th/9803137].

[39] A. Brandhuber, N. Itzhaki, J. Sonnenschein and S. Yankielowicz, *Wilson loops, confinement and phase transitions in large-N gauge theories from supergravity*, *J. High Energy Phys.* **06** (1998) 001 [hep-th/9803263].

[40] Y. Kinar, E. Schreiber and J. Sonnenschein, *Precision 'measurements' of the $q\bar{q}$ potential in mQCD*, *Nucl. Phys.* **B 544** (1999) 633 [hep-th/9809133].

[41] Y. Kinar, E. Schreiber and J. Sonnenschein, *$q\bar{q}$ potential from strings in curved spacetime: classical results*, *Nucl. Phys.* **B 566** (2000) 103 [hep-th/9811192].

[42] J. Sonnenschein, *Wilson loops from supergravity and string theory*, *Class. and Quant. Grav.* **17** (2000) 1257 [hep-th/9910089].

[43] I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: duality cascades and χSB-resolution of naked singularities*, *J. High Energy Phys.* **08** (2000) 052 [hep-th/0007191].

[44] J.M. Maldacena and C. Núñez, *Towards the large-N limit of pure $N = 1$ super Yang-Mills*, *Phys. Rev. Lett.* **86** (2001) 588 [hep-th/0008001].

JHEP07(2004)049

[45] O. Aharony, *The non-AdS/non-CFT correspondence, or three different paths to QCD*, hep-th/0212193.

[46] A. Loewy and J. Sonnenschein, *On the holographic duals of $N = 1$ gauge dynamics*, *J. High Energy Phys.* **08** (2001) 007 [hep-th/0103163].

[47] T. Sakai and J. Sonnenschein, *Probing flavored mesons of confining gauge theories by supergravity*, *J. High Energy Phys.* **09** (2003) 047 [hep-th/0305049].